Gradient flows and PDEs

Olga Turanova

Michigan State University

April 17, 2024

Outline

- Goals:
  - What are gradient flows (GF) on the space of probability measures?
  - How are GFs connected to PDEs?
  - Why is this useful?
- Part I:
  - GF on $\mathbb{R}^d$.
  - GF on space of probability measures.
  - Connection to PDEs.
  - Following: [AGS] "Gradient Flows in Metric Spaces and in the Space of Probability Measures" by Ambrosio, Gigli, Savaré.
- Part II:
  - Recent work on particle methods for nonlinear diffusion equations.
  - Joint work with K. Craig, K. Elamvazhuthi, M. Haberland (2022).

Part I

Gradient flows on Euclidean space

- Let $F : \mathbb{R}^d \to (-\infty, +\infty]$ be λ-convex, some $\lambda \in \mathbb{R}$.
  - This means $F(x) - \frac{\lambda}{2}|x|^2$ is convex.
- Let $x_0 \in \mathbb{R}^d$. Gradient flow of F on $\mathbb{R}^d$ is a curve $x : [0, T] \to \mathbb{R}^d$ such that

$$\begin{cases} x'(t) &= -\nabla F(x(t)) \quad \text{for } t > 0, \\ x(0) &= x_0. \end{cases} \tag{ODE}$$

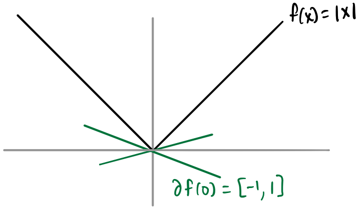
- F is λ-convex $\implies$ ∇F is Lipschitz $\implies$ ODE has a solution for all $x_0 \in \mathbb{R}^d$.

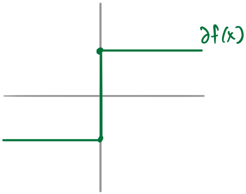
Subdifferential of λ-convex $F : \mathbb{R}^d \to (-\infty, +\infty]$

- We say $p \in \mathbb{R}^d$ is in the subdifferential of F at x, denoted $p \in \partial F(x)$, if

$$F(y) - F(x) \geq (y - x) \cdot p + \frac{\lambda}{2}|x - y|^2 \text{ for all } y \in \mathbb{R}^d.$$

- If x s.t. F is differentiable at x, then $\partial F(x) = \{\nabla F(x)\}$.
- Example ($\lambda = 0$):





- Reformulate (ODE) as,

$$\begin{cases} x'(t) & \in -\partial F(x(t)) \quad \text{for } t > 0, \\ x(0) & = x_0. \end{cases}$$

GF on $\mathcal{P}_2(\mathbb{R}^d)$?

- On Euclidean space: F λ-convex,

$$\begin{cases} x'(t) & \in -\partial F(x(t)) \quad \text{for } t > 0, \\ x(0) & = x_0. \end{cases}$$

- $\mathcal{P}_2(\mathbb{R}^d)$ (probability measures with finite second moment) equipped with W_2 distance.
- For $\mu \ll dx$, we denote its density by μ as well.
- Energies $\mathcal{F} : \mathcal{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ that are proper, lower semicontinous wrt W_2, and λ-convex.
- Let $\mathcal{F} : \mathcal{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ be proper, lower semicontinous wrt W_2, and λ-convex.
- To formulate analogous definition in $\mathcal{P}_2(\mathbb{R}^d)$, need notions of:
  - A curve $\mu(t)$ and its time derivative,
  - Subdifferential of $\mathcal{F}$.

Time derivative of a curve $\mu : [0, T] \to \mathcal{P}_2(\mathbb{R}^d)$

- Definition: metric derivative

$$|\mu'|(t) := \lim_{s \to t} \frac{W_2(\mu(t), \mu(s))}{|t - s|}.$$

- We consider curves $\mu \in AC^2([0, T]; \mathcal{P}_2(\mathbb{R}^d))$; for such curves this limit is well-defined.
- We use $\mu(t)$ to denote $\mu(\cdot, t)$.
- $|\mu'|(t) \in \mathbb{R}$; not quite analogous to $x'(t) \in \mathbb{R}^d$.
- Theorem [AGS 8.3.1]: $|\mu'|(t)$ is the metric derivative of μ "if and only if"

$$\partial_t \mu + \nabla \cdot (\boldsymbol{v}\mu) = 0 \text{ weakly on } [0, T] \times \mathbb{R}^d$$

  holds for a velocity field $\boldsymbol{v} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\boldsymbol{v}(t, \cdot) \in L^2(\mu(t); \mathbb{R}^d)$, with

$$|\mu'(t)| = \|\boldsymbol{v}\|_{L^2(\mu(t); \mathbb{R}^d)} \text{ a.e. } t.$$

- $\boldsymbol{v}$ will be our analogy of x'.

Subdifferential of λ-convex $\mathcal{F}$

- Definition: Let $\mu \in D(\mathcal{F})$. We say $\boldsymbol{\xi} : \mathbb{R}^d \to \mathbb{R}^d$ with $\boldsymbol{\xi} \in L^2(\mu; \mathbb{R}^d)$ is in the subdifferential of $\mathcal{F}$ at μ if for all $\nu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\mathcal{F}(\nu) - \mathcal{F}(\mu) \geq \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \boldsymbol{\xi}(x), y - x \rangle d\boldsymbol{\gamma}(x, y) + \frac{\lambda}{2} W_2^2(\mu, \nu)$$

  for all $\boldsymbol{\gamma} \in \Gamma_0(\mu, \nu)$. We write $\boldsymbol{\xi} \in \partial \mathcal{F}(\mu)$.
- Attempt 1: Let $\boldsymbol{v} \in L^2(\mu; \mathbb{R}^d)$ be "time derivate of μ" as on previous slide. How about

$$\boldsymbol{v}(t) \in -\partial \mathcal{F}(\mu(t))?$$

- Not quite: there could be tons of stuff in the subdifferential.
- Theorem [AGS 10.1.5]: For "nice" $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, the subdifferential $\partial \mathcal{F}(\mu)$ has an element of minimal norm ($\| \cdot \|_{L^2(\mu; \mathbb{R}^d)}$). This element is denoted $\partial^\circ \mathcal{F}(\mu)$.

GF of λ-convex $\mathcal{F}$

- Definition [AGS 11.1.1]: We say $\mu \in AC^2([0,T], \mathcal{P}_2(\mathbb{R}^d))$ is a gradient flow of $\mathcal{F}$ if $\mu(t)$ solves
$$\partial_t \mu + \nabla \cdot (\boldsymbol{v}\mu) = 0, \tag{1}$$
and,
$$\boldsymbol{v}(t) = -\partial^\circ \mathcal{F}(\mu(t)) \text{ a.e. } t \in [0,T].$$

- Theorem [AGS 11.2.1]: Suppose $\mathcal{F} \colon \mathcal{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ is proper, lsc, and λ-convex, and $\mu^0 \in \overline{D(\mathcal{F})}$.
Then, there exists a unique gradient flow $\mu(t)$ of $\mathcal{F}$ with initial condition μ^0.
- The continuity equation (1) will connect GF to PDEs, once we characterize $-\partial^\circ \mathcal{F}(\mu(t))$ explicitly.

$\partial^\circ \mathcal{F}(\mu)$ and PDEs

- Formally (for twice differentiable μ, nice $\mathcal{F}$):

$$\mathcal{F}(\mu) = \int F(x, \mu(x), \nabla \mu(x))\, dx \implies \partial^\circ \mathcal{F}(\mu) = \nabla \frac{\delta \mathcal{F}}{\delta \mu},$$

where $\frac{\delta \mathcal{F}}{\delta x}$ is the first variation of $\mathcal{F}$.

- More generally: V convex, W convex and even, f convex (and more):

$\mathcal{F}$	$\partial^\circ \mathcal{F}(\mu)$	PDE
$\mathcal{V}(\mu) = \int V(x)\, d\mu$	∇V	$\partial_t \mu - \operatorname{div}(\mu \nabla V) = 0$
$\mathcal{W}(\mu) = \int W(x-y) d\mu(x) d\mu(y)$	$\nabla W * \mu$	$\partial_t \mu - \operatorname{div}(\mu \nabla W * \mu) = 0$
$\mathcal{E}(\mu) = \int f(\mu)\, dx$	$\nabla f'(\mu)$	$\partial_t \mu - \nabla(\mu \nabla f'(\mu)) = 0$

- Subexamples:
  - $f(s) = s \log s - s \leftrightsquigarrow$ heat equation.
  - $f(s) = \frac{1}{m-1} s^m \leftrightsquigarrow$ porous medium eq'n ($m > 1$), fast diffusion ($\frac{d}{d+2} < m < 1$).
- $\mathcal{F} := \mathcal{E} + \mathcal{V} + \mathcal{W}$, say, yields corresponding drift-diffusion-aggregation PDE.

Curve of maximal slope: Euclidean space

- Let $y(t)$ be any curve.

$$F(y(0)) - F(y(t)) = \int_0^t -\frac{d}{dr}F(y(r))\,dr = \int_0^t -\nabla F(y(r))y'(r)\,dr$$
$$\leq \frac{1}{2}\int_0^t |\nabla F(y(r))|^2\,dr + \frac{1}{2}\int_0^t |y'(r)|^2\,dr,$$

with equality holding $\iff$ $y'(r) = -\nabla F(y(r))$ for a.e. r.

- So $x(t)$ is GF of F $\iff$ equality holds $\iff$ $\geq$ holds $\iff$

$$F(x(0)) - F(x(t)) \geq \frac{1}{2}\int_0^t \int_0^t |\nabla F(x(r))|^2\,dr + \frac{1}{2}\int_0^t |x'(r)|^2\,dr.$$

- Theorem [AGS 11.2.1]: For "nice" $\mathcal{F}$, μ is the gradient flow of $\mathcal{F}$ $\iff$ μ is a *Curve of Maximal Slope*; i.e., for all $t \in [0, T]$,

$$\mathcal{F}(\mu(0)) - \mathcal{F}(\mu(t)) \geq \frac{1}{2}\int_0^t |\partial\mathcal{F}|^2(\mu(r))dr + \frac{1}{2}\int_0^t |\mu'|^2(r)dr.$$

  - Here $|\partial\mathcal{F}|(\mu(r))$ is the *metric slope* of $\mathcal{F}$ at $\mu(r)$.
  - We can think $|\partial\mathcal{F}|(\mu) = \|\partial^\circ\mathcal{F}(\mu)\|_{L^2(\mu;\mathbb{R}^d)}$.

JKO (Jordan-Kinderleher-Otto) scheme

$$x'(t) = -\nabla F(x), \quad x(0) = x_0.$$

- Fix small time step $\tau > 0$, let $x_0^\tau = x_0$, and define, for $k \geq 1$,

$$x_{k+1}^\tau = \mathrm{argmin}\left(F(x) + \frac{|x - x_k^\tau|^2}{2\tau} \right),$$

  which implies $-\nabla F(x_{k+1}^\tau) = \dfrac{x_{k+1}^\tau - x_k^\tau}{\tau}$ holds.
- The x_k^τ converge to $x(t)$ as $\tau \to 0$ (implicit Euler scheme.)
- "JKO scheme" for $\mathcal{F}$: given initial data μ_0, define, for $k \geq 1$,

$$\mu_{k+1}^\tau = \mathrm{argmin}\left(\mathcal{F}(\mu) + \frac{W_2(\mu, \mu_k^\tau)^2}{2\tau} \right).$$

- Theorem [AGS 11.2.1]: For "nice" $\mathcal{F}$, there exists a limiting μ, and it's a GF of $\mathcal{F}$ in the previous sense.

Remark

- We've seen three equivalent (for "nice" $\mathcal{F}$) formulations of GFs on $\mathcal{P}_2(\mathbb{R}^d)$:
  - Pointwise-differential formulation
  - Curve of maximal slope
  - Minimizing movement scheme.
- There are others!

Part II

Introduction

- Fix $\Omega \subset \mathbb{R}^d$ (convex) and $\bar{\rho} : \Omega \to \mathbb{R}^+$ (log-concave), nice.
- We focus on:

$$\begin{cases} \partial_t \rho - \text{div}\left(\rho \nabla \left(\frac{\rho}{\bar{\rho}}\right)\right) = 0 \text{ in } \Omega \times (0, \infty), \\ \partial_\nu \left(\frac{\rho}{\bar{\rho}}\right) = 0 \text{ on } (\partial\Omega) \times (0, \infty). \end{cases} \tag{D}$$

- If $\bar{\rho} \equiv 1$, then the diffusion term becomes,

$$\text{div}(\rho \nabla \rho) = \frac{1}{2} \text{div}(\nabla \rho^2) = \frac{1}{2} \Delta(\rho^2).$$

  So this PDE is an inhomogeneous porous medium equation.
- Today: deterministic particle method.
- Challenge: $\rho(0) = \frac{1}{N} \sum_{i=1}^N \delta_{X_i^0} \not\Rightarrow \rho(t) = \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)}$.

Introduction

- Part I stuff implies:

$$\partial_t \rho - \text{div}\left(\rho \nabla \left(\frac{\rho}{\bar{\rho}}\right)\right) = 0 \quad \Longleftrightarrow \quad \rho \text{ GF of } \mathcal{E}[\rho] = \frac{1}{2}\int \frac{\rho}{\bar{\rho}}\, d\rho.$$

- Main idea:

  regularize $\mathcal{E}$

  in a way that ensures "particles remain particles."

- We define the regularized energy

$$\mathcal{E}_\varepsilon[\rho] = \frac{1}{2}\int \frac{(\rho * \zeta_\varepsilon)^2}{\bar{\rho}}\, dx,$$

  where ζ_ε is a mollifier "with enough decay."

- This idea originates from Lions, Mas-Gallic (2001): study PME, $\bar{\rho} \equiv 1$.
- Carillo, Craig, Patacchini (2019): $\bar{\rho} \equiv 1$ and with aggregation and drift.
- Related work: Oelschläger (1990); Lu, Slepčev, Wang (2023); Carrillo, Esposito, Wu (2023); Craig, Jacobs, T. (2023).

On the convex domain $\Omega \subset \mathbb{R}^d$

$$\begin{cases} \partial_t \rho - \operatorname{div}\left(\rho \nabla \left(\frac{\rho}{\bar{\rho}}\right)\right) = 0 \text{ in } \Omega \times (0, \infty), \\ \partial_\nu \left(\frac{\rho}{\bar{\rho}}\right) = 0 \text{ on } (\partial\Omega) \times (0, \infty). \end{cases} \tag{D}$$

- Solutions to (D) are Wasserstein gradient flows of the energy:

$$\mathcal{F}[\rho] = \mathcal{E}[\rho] + \mathcal{V}_\Omega[\rho], \quad \text{where } \mathcal{V}_\Omega[\rho] = \begin{cases} 0 \text{ if } \operatorname{supp} \rho \subset \bar{\Omega}, \\ +\infty \text{ otherwise.} \end{cases}$$

- We also approximate $\mathcal{V}_\Omega$ via a "soft cutoff potential":

$$\mathcal{V}_k[\rho] := \int_{\mathbb{R}^d} V_k \, d\rho, \quad \text{where } \begin{cases} V_k(x) = 0 & \text{for all } x \in \bar{\Omega}, \text{ for all } k, \\ V_k(x) \to +\infty & \text{as } k \to \infty \text{ for } x \in \bar{\Omega}^c. \end{cases}$$

- Main result: take $\varepsilon \to 0$ and $k \to \infty$ in

$$\mathcal{F}_{\varepsilon,k} = \mathcal{E}_\varepsilon + \mathcal{V}_k.$$

The approximation and particle data

Lemma: GFs of $\mathcal{F}_{\varepsilon,k}$ are well-defined and are distributional solutions of

$$\partial_t \rho - \text{div}\left(\rho \nabla \left(\zeta_\varepsilon * \left(\frac{\zeta_\varepsilon * \rho}{\bar{\rho}}\right) + V_k\right)\right) = 0.$$

Lemma: Let $\rho^N_{\varepsilon,k}$ denote the GF of $\mathcal{F}_{\varepsilon,k}$ with initial data $\frac{1}{N}\sum_{i=1}^N \delta_{X_i^0}$. We have, for $t > 0$,

$$\rho^N_{\varepsilon,k}(x,t) = \frac{1}{N}\sum_{i=1}^N \delta_{X_i(t)},$$

where the X_i evolve via the system of ODEs associated to the continuity equation:

$$\frac{d}{dt}X_j(t) = -V_\varepsilon^{(j)}(X_1(t), ..., X_N(t)), \quad X_j(0) = X_j^0,$$

where V_ε is given by,

$$V_\varepsilon^{(i)}(y_1, ..., y_N) = \sum_{j=1}^N m_j \int_{\mathbb{R}^d} \nabla\zeta_\varepsilon(y_i - z)\zeta_\varepsilon(z - y_j)\frac{1}{\bar{\rho}(z)}\, dz - \nabla V_k(y_i).$$

Numerical method

- Discretize initial data into particles: $\rho_0 \approx \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i^0}$.
- Let the particles evolve via the ODE system:

$$\frac{d}{dt}X_j(t) = -V_\varepsilon^{(j)}(X_1(t), ..., X_N(t)), \quad X_j(0) = X_j^0.$$

- Let

$$\rho_{\varepsilon,k}^N(x,t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i(t)}.$$

- Consider "blobs" over each particle:

$$\tilde{\rho}_{\varepsilon,k}^N(x,t) = \frac{1}{N}\sum_{i=1}^{N} \zeta_\varepsilon(x - X_i(t)) = (\zeta_\varepsilon * \rho_{\varepsilon,k}^N)(x,t).$$

- This $\tilde{\rho}_{\varepsilon,k}^N$ is our approximate solution.

Application to sampling

$$\partial_t \rho - \operatorname{div}\left(\rho \nabla \left(\frac{\rho}{\bar{\rho}}\right)\right) = 0 \text{ in } \Omega \times (0,\infty), \quad \partial_\nu \left(\frac{\rho}{\bar{\rho}}\right) = 0 \text{ on } (\partial\Omega) \times (0,\infty).$$

- Fact: $W_2(\rho(t), \mathbb{1}_{\overline{\Omega}}\bar{\rho}) \to 0$ as $t \to \infty$.
- Given
  - target probability measure $\bar{\rho}$,
  - reference $\rho_0 = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i^0}$.

  Goal: find $\rho_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i}$ approximating $\bar{\rho}$.
- We prove, under the same assumptions as the main theorems:

Corollary

There exist $k = k(t) \to +\infty$, $\varepsilon = \varepsilon(k) \to 0$, and $N = N(\varepsilon) \to +\infty$ so that

$$\lim_{t \to +\infty} W_1\left(\rho^N_{\varepsilon,k}(\cdot,t), \mathbb{1}_{\overline{\Omega}}\bar{\rho}\right) = 0.$$

- So, Corollary gives a way to approximate $\bar{\rho}$ by an empirical measure.

Numerical method

Videos!

Convergence of numerical method

Assumptions:

1. $\Omega \subset \mathbb{R}^d$ is convex;
2. $\bar{\rho} \in C^1(\Omega)$ is bounded from above and away from zero on Ω;
3. $\bar{\rho}$ log-concave on Ω, i.e $\bar{\rho}(x) = e^{\Phi(x)}$ for some Φ that's concave on Ω.
4. mild assumptions on initial data ρ_0.

Theorem (Craig, Elamvazhuthi, Haberland, T (2023))

Let ρ solve the PDE (D) with initial data ρ_0. Then as $k \to \infty$, there exist subsequences $\varepsilon(k) \to 0$ and $N(k) \to \infty$, such that,

$$\rho^N_{\varepsilon,k}(t) \to \rho(t) \textit{ and } \tilde{\rho}^N_{\varepsilon,k}(t) \to \rho(t),$$

in the 1-Wassertein distance, uniformly in $t \in [0, T]$.*

(*) hence also in the sense of "integrating against test functions"

Results on GF

Recall

$$\mathcal{F} = \mathcal{E} + \mathcal{V}_\Omega, \quad \mathcal{F}_{\varepsilon,k} = \mathcal{E}_\varepsilon + \mathcal{V}_k.$$

Under the same assumptions:

Theorem (Craig, Elamvazhuthi, Haberland, T (2023))

Let $\rho_{\varepsilon,k}$ be the Wasserstein gradient flow of $\mathcal{F}_{\varepsilon,k}$ with initial data ρ_0. Then as $k \to \infty$, there exists a subsequence $\varepsilon(k) \to 0$ such that such that,

$$\rho_{\varepsilon,k}(t) \to \rho(t),$$

in the 1-Wassertein distance, uniformly in $t \in [0, T]$, where ρ is the gradient flow of $\mathcal{F}$ with initial data ρ_0.

- The previously stated result follows from this.

Elements of the proof

1. AGS: Wasserstein GFs for $\mathcal{F} = \mathcal{E} + \mathcal{V}_\Omega$ are well-defined.
2. We show: GF for $\mathcal{F}_{\varepsilon,k} = \mathcal{E}_\varepsilon + \mathcal{V}_k$ are well-defined.
   - $\mathcal{E}_\varepsilon$ is λ_ε-convex along generalized geodesics in Wasserstein space (where $\lambda_\varepsilon < 0$ and $\lambda_\varepsilon \to -\infty$ as $\varepsilon \to 0$).
   - does not require log-concavity assumption.
3. Γ-convergence of $\mathcal{F}_{\varepsilon,k}$ to $\mathcal{F}$.
4. Obtain estimate on an "H^1-like norm" for GFs of $\mathcal{F}_{\varepsilon,k}$ that is uniform in ε.
   - key ingredient: energy estimate for the ε-PDE.
5. Γ-convergence of metric slopes of $\mathcal{F}_{\varepsilon,k}$ to those of $\mathcal{F}$.
6. Conclude via a general result (variant of Serfaty, 2011) on convergence of GF:

$$\underbrace{\mathcal{F}_{\varepsilon,k}(\rho_{\varepsilon,k}(0)) - \mathcal{F}_{\varepsilon,k}(\rho_{\varepsilon,k}(t))}_{\text{use step 3}} \geq \frac{1}{2}\underbrace{\int_0^t |\rho'_{\varepsilon,k}|^2(r)dr}_{\text{use compactness}} + \underbrace{\frac{1}{2}\int_0^t |\partial\mathcal{F}_{\varepsilon,k}|^2(\rho_{\varepsilon,k}(r))dr}_{\text{use step 5}}.$$

More on the proof: H^1-like bound

Lemma

Suppose ρ is a solution to

$$\partial_t \rho - \operatorname{div}\left(\rho \nabla \left(\zeta_\varepsilon * \left(\frac{\zeta_\varepsilon * \rho}{\bar{\rho}}\right)\right)\right) = 0.$$

Then we have,

$$\int_0^T \int |\nabla \rho * \zeta_\varepsilon|^2 \, dx \, dt \le C.$$

Follows from energy estimate:

$$\frac{d}{dt} \int \rho \log(\rho) \, dx + \int |\nabla \rho * \zeta_\varepsilon|^2 = \langle \nabla \rho * \zeta_\varepsilon, (\rho * \zeta_\varepsilon) \nabla \left(\frac{1}{\bar{\rho}}\right) \rangle.$$

Energy estimate with $\bar{\rho} \equiv 1$:

$$\partial_t \rho - \operatorname{div}(\rho(\nabla\rho * \zeta_\varepsilon * \zeta_\varepsilon)) = 0.$$

Note: $\frac{d}{dt}\int \rho\log(\rho)\,dx = \frac{d}{dt}\left(\int \rho\log(\rho)\,dx - \int \rho\,dx\right)$, so that,

$$\begin{aligned}
\frac{d}{dt}\int \rho\log(\rho)\,dx &= \int \rho_t\log(\rho) + \rho\frac{\rho_t}{\rho} - \rho_t\,dx \\
&= \int \operatorname{div}(\rho(\nabla\rho * \zeta_\varepsilon * \zeta_\varepsilon))\log(\rho)\,dx \\
&= -\int \rho(\nabla\rho * \zeta_\varepsilon * \zeta_\varepsilon)\nabla\log(\rho)\,dx \\
&= -\int \rho(\nabla\rho * \zeta_\varepsilon * \zeta_\varepsilon)\frac{\nabla\rho}{\rho}\,dx \\
&= -\int (\nabla\rho * \zeta_\varepsilon * \zeta_\varepsilon)\nabla\rho\,dx = -\int (\nabla\rho * \zeta_\varepsilon)^2\,dx.
\end{aligned}$$

Recall: if ζ is even, then

$$\int (f * \zeta)(x)g(x)\,dx = \int f(x)(\zeta * g)(x)\,dx.$$

Thank you!