HV Geometry for Signed Signal Comparison

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Problem: Signed Signal Comparison

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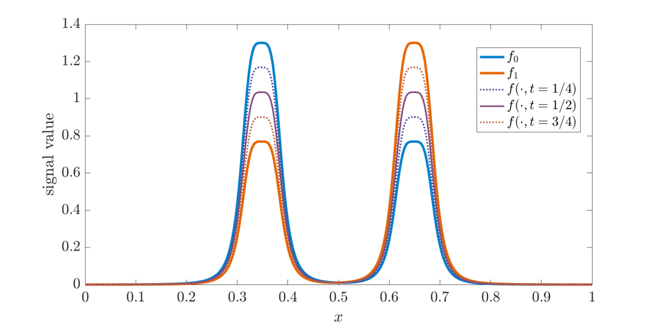
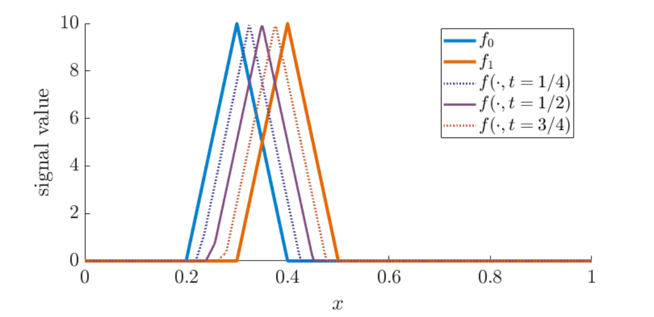
Metric Candidate, L^2 distance: $\left(\int_0^1 |f_0(x) - f_1(x)|^2 dx\right)^{\frac{1}{2}}$



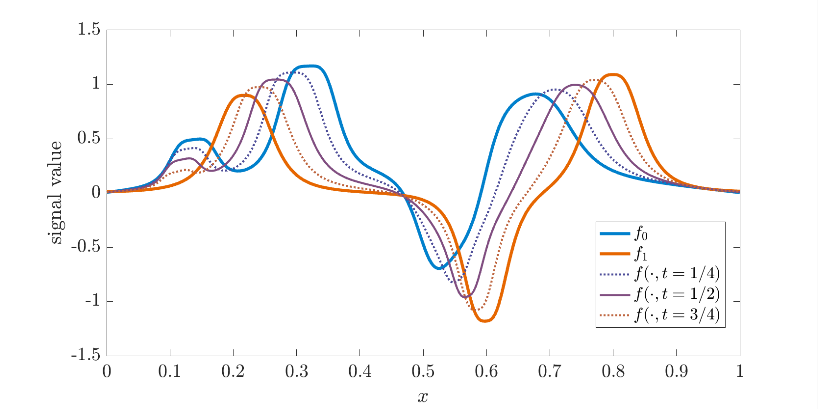
Vertical Deformation

Metric Candidate, Wasserstein distance: horizontal deformation



W_2 distance is small, but L^2 distance is large.

Want: Horizontal and Vertical Deformation



The computed geodesic in the space of signals based on the HV geometry.

New Metric: H(orizontal)V(ertical) Geometry

Revisit the Benamou–Brenier formulation for W_2

The so-called dynamic formulation of optimal transport:

$$W_2^2(f_0, f_1) = \min_{(v,f)} \int_0^1 \int_0^1 v^2 f \, dx dt.$$

subject to the constraints for all admissible paths

$$\begin{aligned} \partial_t f &= -\operatorname{div}(f\, v) \\ f(\cdot, t=0) = f_0,\ \ & f(\cdot, t=1) = f_1. \end{aligned} \tag{1}$$

Requires $\int f_0 dx = \int f_1 dx$ and $f_0, f_1 \geq 0$.

[Benamou–Brenier, 2000]

A Metric Induced by the HV Geometry

Given a finite interval, e.g., $[0,1]$, consider $f_0, f_1 \in L^2(0,1)$ with all the admissible paths satisfying

$$\partial_t f = \boxed{-\partial_x f \cdot v} + z \quad \text{on } [0,1]\times[0,1],$$
$$v(0,\cdot) = v(1,\cdot) = 0, \quad f(\cdot,0) = f_0, \;\; f(\cdot,1) = f_1.$$

[Ambrosio and Crippa, 2014]

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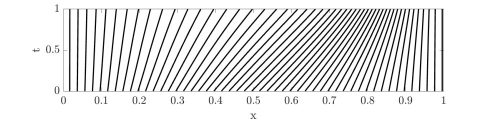
$$f(\Phi(x,t), t) = f_0(x) + \int_0^t z(\Phi(x,s), s)\, ds$$

where Φ is the flow of the vector field v:

$$\partial_t \Phi(x,t) = v(\Phi(x,t), t), \quad \Phi(x, 0) = x.$$



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Define

$$\mathcal{A}(f_0, f_1) := \big\{ (f, v, z) \text{ satisfies } (2) \big\}.$$

[Miller-Younes, 2001], [Trouvé-Younes, 2005], [H., Slepčev and Yang, 2023]

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$$\mathcal{A}(f_0, f_1) := \big\{ (f, v, z) \text{ satisfies } (2) \big\}.$$

For $\kappa > 0$, $\lambda \geq 0$, $\varepsilon > 0$, define:

$$d_{HV(\kappa,\lambda,\varepsilon)}(f_0, f_1) := \inf_{(f,v,z)\in\mathcal{A}(f_0,f_1)} \sqrt{A_{\kappa,\lambda,\varepsilon}(f, v, z)}, \text{ where}$$

$$A_{\kappa,\lambda,\varepsilon}(f, v, z) = \frac{1}{2}\int_0^1\int_0^1 \left(\kappa v^2 + \lambda v_x^2 + \varepsilon v_{xx}^2 + z^2\right) \, dxdt\,.$$

[Miller-Younes, 2001], [Trouvé-Younes, 2005], [H., Slepčev and Yang, 2023]

Degeneracy Without the Second Derivatives

But why not more naturally take

$$A_{\kappa,\lambda}(f, v, z) = \frac{1}{2}\int_0^1 \int_0^1 \left(\kappa v^2 + \lambda v_x^2 + z^2\right)\, dxdt\,.$$

[H., Slepčev and Yang, 2023]

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Proposition

*If $\varepsilon = 0$, there exists $H > 0$ such that there is **no path** between $f_0 \equiv 0$ and $f_1 \equiv H$ minimizing the action.*

[H., Slepčev and Yang, 2023]

Properties of d_{HV}

d_{HV} is complete on $L^2(0, 1)$ and admits geodesics.

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1. Let $\{f_n\}_{n\in\mathbb{N}} \subseteq L^2(0,1)$. If $f_n \to f$ in d_{HV}, then $f_n \to f$ in L^2.

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2. (Regularity) If $f_0, f_1 \in H^1$, then any action minimizing path $f \in L^\infty(0,1,H^1(0,1))$.
3. (Stability) Assume $f_0^n, f_1^n \in L^2(0,1)$ for all $n \in \mathbb{N}$, $f_0^n \to f_0, \;\; f_1^n \to f_1$ in $L^2(0,1)$ as $n \to \infty$.
Let $(f^n, v^n, z^n) \in \mathcal{A}(f_0^n, f_1^n)$ be action minimizing paths. Then there exists (f, v, z) such that along a subsequence

$$f^n \to f, \quad z^n \to z, \quad v^n \to v$$

Furthermore, (f, v, z) is an action minimizing path between f_0 and f_1.

[H., Slepčev and Yang, 2023]

Numerical Scheme: Iterating Between Two Steps

From $(f_{old}, v_{old}, z_{old})$ to $(f_{new}, v_{new}, z_{new})$:

[H., Slepčev and Yang, 2023]

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From $(f_{old}, v_{old}, z_{old})$ to $(f_{new}, v_{new}, z_{new})$:
Step 1: $(f_{new}, \tilde{z}) = \mathcal{G}_1(v_{old})$, from v to (f, z).
minimizing the objective functional:

$$\min_{f,z} \frac{1}{2} \int_0^1 \int_0^1 z^2 \, dxdt, \quad \text{s.t.} \quad (f, v, z) \in \mathcal{A}.$$

(f, z) has analytic formulation given v.

[H., Slepčev and Yang, 2023]

Step 2: $(v_{new}, z_{new}) = \mathcal{G}_2(f_{new})$, from f to (v, z) minimizing the functional:

$$\min_{v,z} \frac{1}{2} \int_0^1 \int_0^1 \kappa v^2 + \lambda v_x^2 + \varepsilon v_{xx}^2 + z^2 \, dxdt, \quad \text{s.t. } (f, v, z) \in \mathcal{A}.$$

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v obtained by solving a fourth order boundary value problem,

$$\varepsilon v_{xxxx} - \lambda v_{xx} + \kappa v + z f_x = 0 \text{ on } (0,1)^2$$
$$v = 0 \text{ and } v_{xx} = 0 \text{ on } \{0,1\} \times [0,1]$$

$z = f_t + v f_x$ given by constraint.

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Remark: (v, z) can be viewed as “tangent vector”.

[H., Slepčev and Yang, 2023]

Iterating Between These Two Steps

$$A_{\kappa,\lambda,\varepsilon}(f^{\text{new}}, v^{\text{new}}, z^{\text{new}}) \underbrace{\leq}_{(v^{\text{new}}, z^{\text{new}}) = \mathcal{G}_2(f), f^{\text{new}}=f} A_{\kappa,\lambda,\varepsilon}(f, v^{\text{old}}, z) \underbrace{\leq}_{(f,z) = \mathcal{G}_1(v^{\text{old}})} A_{\kappa,\lambda,\varepsilon}(f^{\text{old}}, v^{\text{old}}, z^{\text{old}}),$$

[H., Slepčev and Yang, 2023]

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1: Given $(f^{(0)}, v^{(0)}, z^{(0)}) \in \mathcal{A}$, max iterations N, tolerance $\delta > 0$.
2: **for** $n = 1$ to N **do**
3: Compute $(\tilde{f}, \tilde{z}) = \mathcal{G}_1(v^{(n)})$ with $\mathcal{G}_1$ and set $f^{(n+1)} = \tilde{f}$.
4: Set $(v^{(n+1)}, z^{(n+1)}) = \mathcal{G}_2(f^{(n+1)})$ with $\mathcal{G}_2$
5: **if** $|A_{\kappa,\lambda,\varepsilon}(f^{(n+1)}, v^{(n+1)}, z^{(n+1)}) - A_{\kappa,\lambda,\varepsilon}(f^{(n)}, v^{(n)}, z^{(n)})| < \delta$ **then**
6: Return $\left(f^{(n+1)}, z^{(n+1)}, v^{(n+1)}\right)$; **Break**.
7: **end if**
8: **end for**

[H., Slepčev and Yang, 2023]

Initialization Selection

We propose two different types of initial guesses.

1. **Zero-velocity initialization.** Set $v^{(0)}(x,t) \equiv 0$, and

$$\Rightarrow f^{(0)}(x,t) = (1-t)f_0(x) + tf_1(x), \quad z^{(0)}(x,t) = f_1(x) - f_0(x)$$

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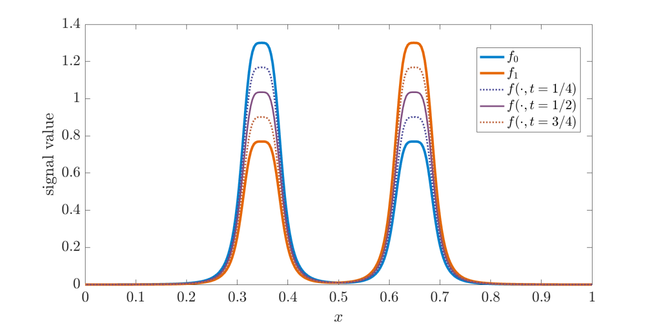
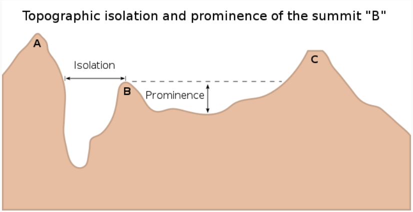


[H., Slepčev and Yang, 2023]

2. **Prominence-matching initialization.**
Let k be a positive integer. For the given f_0 and f_1, we each select k local maxima with the largest k prominence.



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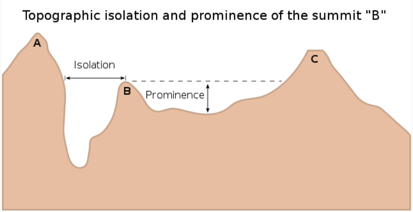

The location of the local maxima are denoted by $\{x_i\}$ and $\{y_i\}$, $1 \leq i \leq k$, respectively.

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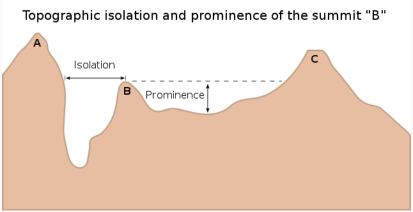
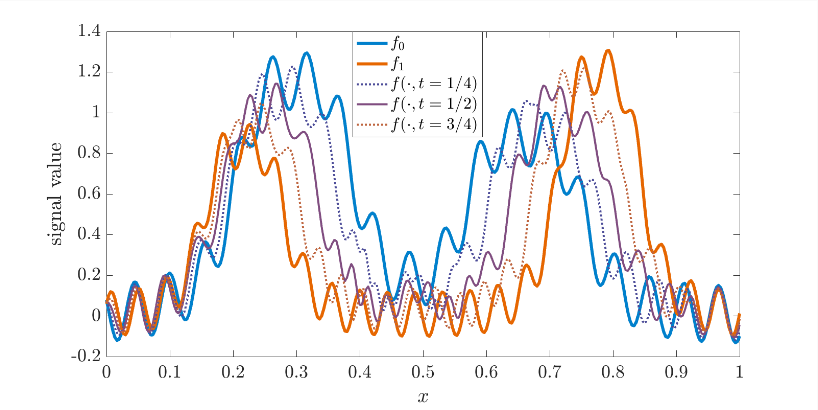


The location of the local maxima are denoted by $\{x_i\}$ and $\{y_i\}$, $1 \leq i \leq k$, respectively.

Construct a piecewise linear map T such that $T(x_i) = y_i$, $T(0) = 0, T(1) = 1$.

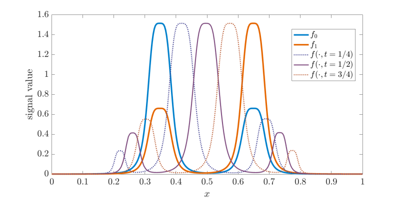
[H., Slepčev and Yang, 2023], [Wikipedia]

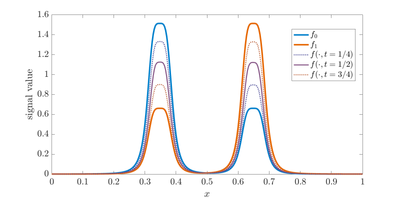


Example using prominence-matching initialization

[H., Slepčev and Yang, 2023]

Numerical Results: Non-Uniqueness of the Minimizing Path

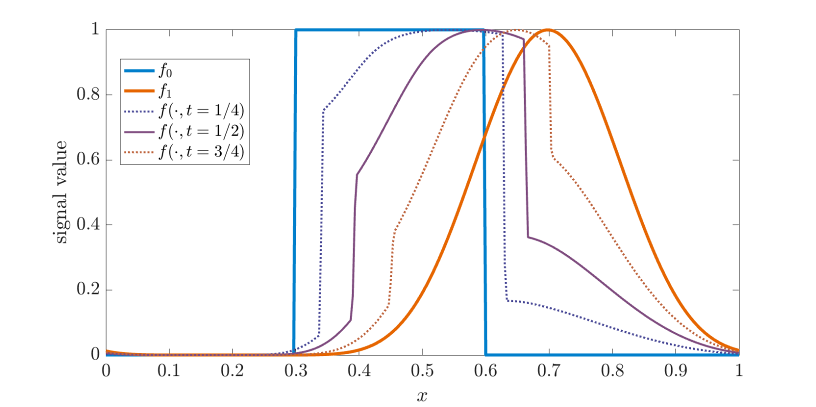




For appropriate ratio of bump heights, both dominant transport mechanisms produce the same action.

[H., Slepčev and Yang, 2023]

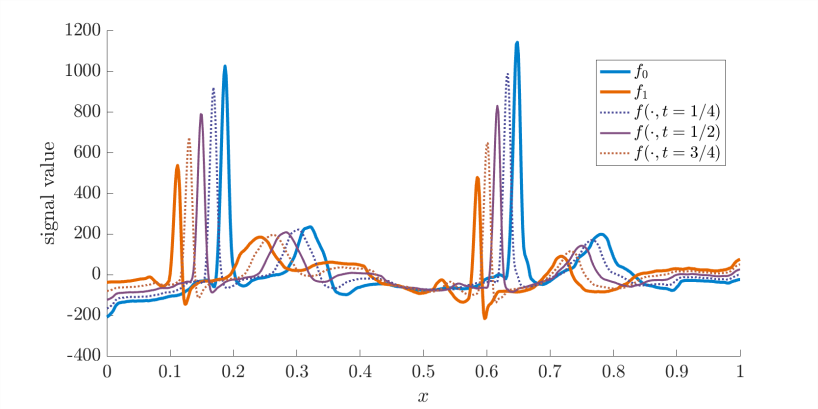
Numerical Results: Non-Smooth Signals



Algorithm allows for non-smooth data.

[H., Slepčev and Yang, 2023]

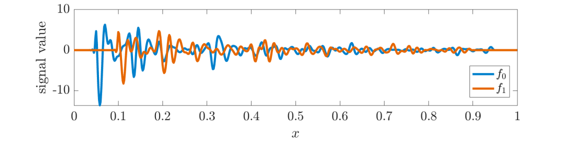
Numerical Results: Electrocardiography (ECG) Signals



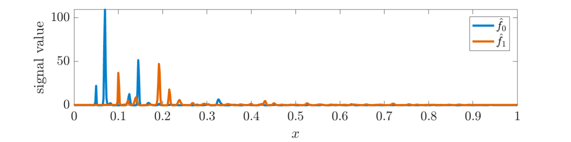
The large features (peaks) are matched via horizontal transport.

[H., Slepčev and Yang, 2023]

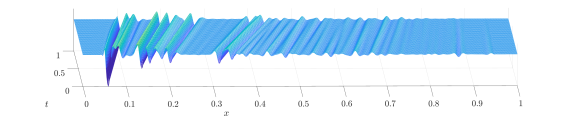
Numerical Results: Seismic Signals



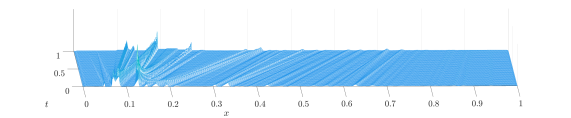
Signals



Normalized Signals



HV geodesic



OT geodesic

Acknowledgments

Thank Katy Craig, Dejan Slepčev and Yunan Yang!

Thank all the workshop organizers!

Thanks for listening!

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Proposition

If $\varepsilon = 0$, there exists $H > 0$ such that there is ***no path*** *between $f_0 \equiv 0$ and $f_1 \equiv H$ minimizing the action.*

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Lemma

If $\varepsilon = 0$, then for all $\lambda \in [0,\infty)$ there exists $H \in \mathbb{R}$ such that the linear interpolation between $f_0 \equiv 0$ and $f_1 \equiv H$ is not optimal.

The optimal path has $v \not\equiv 0$.

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The optimal path has $v \not\equiv 0$.
If there existed an optimizing path (f,v,z), one could construct path of lower action by creating two copies of f shrank to interval $\frac{1}{2}$. The velocity is reduced to one-half.

[H., Slepčev and Yang, 2023]

Stability of d_{HV}: Precise statement

(Stability) Let $f_0, f_1 \in L^2(0,1)$.
Assume $f_0^n, f_1^n \in L^2(0,1)$ for all $n \in \mathbb{N}$, $f_0^n \to f_0, \ \ f_1^n \to f_1$ in $L^2(0,1)$ as $n \to \infty$.
Let $(f^n, v^n, z^n) \in \mathcal{A}(f_0^n, f_1^n)$ be action minimizing paths. Then there exists $(f, v, z) \in \mathcal{A}(f_0, f_1)$ such that along a subsequence

$$
\begin{aligned}
f^n &\overset{*}{\rightharpoonup} f \quad \text{in } L^\infty((0,1), L^2(0,1)) \\
f^n &\to f \quad \text{in } C((0,1), (L^2(0,1), d_{HV})) \\
z^n &\rightharpoonup z \quad \text{in } L^2((0,1), L^2(0,1)) \\
v^n &\rightharpoonup v \quad \text{in } L^2([0,1]; H^2(0,1)).
\end{aligned}
$$

Furthermore (f, v, z) is an action minimizing path between f_0 and f_1.

[H., Slepčev and Yang, 2023]

Parameter Selection

We suggest

$$\kappa = 0.01\frac{H^2}{L^2}, \quad \lambda = 0.02H^2, \text{ and } \varepsilon = 0.2H^2W^2.$$

where

H is the average vertical variation in the data;

W is the typical width of features in the data;

L is the maximum horizontal distance between the features to be matched.

A suggestion for H is the L^2 distance between the signals.

[H., Slepčev and Yang, 2023]

Scaling Properties of d_{HV}

Proposition

Consider $f_0, f_1 \in L^2(0,1)$. Let $c > 0$. Then

(i) $d_{HV}(f_0 + c, f_1 + c) = d_{HV}(f_0, f_1)$

(ii) $d_{HV(c^2\kappa, c^2\lambda, c^2\varepsilon)}(cf_0, cf_1) = c d_{HV(\kappa,\lambda,\varepsilon)}(f_0, f_1)$

To indicate the behavior of the action with respect to rescaling the space extend f_0 and f_1 periodically to $\mathbb{R}$. Likewise, given a path (f, v, z) consider it extended periodically to $\mathbb{R}$. Then for $L \in \mathbb{N}$,

(iii) $A_{L^2\kappa,\lambda,\varepsilon/L^2}(f(L\,\cdot\,,\,\cdot\,), v(L\,\cdot\,,\,\cdot\,), z(L\,\cdot\,,\,\cdot\,)) = A_{\kappa,\lambda,\varepsilon}(f, v, z)$, *where the action is considered only on $[0,1]$, as usual.*

[H., Slepčev and Yang, 2023]