The Stefan problem via optimal stopping times

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June 26, 2022

Stefan problem: freezing

The supercooled Stefan problem describes freezing of the supercooled water into ice.

"The freezing point of water is 0°C but it might be more accurate to say that the melting point of ice is 0°C. This is because, for a number of complex reasons, water exists in liquid form well below 0°C... Two of the factors influencing the freezing of supercooled droplets are the need for a freezing nuclei (usually ice crystals) and latent heat which is released when water freezes" - from SKYbrary Wiki

Formally, it can be written in divergence form as

$$(St_1) \qquad (w + \chi_{\{w<0\}})_t - \Delta w = 0.$$

Here w denotes the temperature of the supercooled water, and the set $\{w < 0\}$ denotes the region occupied by the water. In view of water as brownian particles, we denote $u := -w$, for which the equation becomes

$$(St_1). \qquad (u - \chi_{\{u>0\}})_t - \Delta u = 0.$$

Stefan problem: melting

In contrast, the Stefan problem for melting ice into water can be written as

$$(St_2) \qquad (u+\chi_{\{u>0\}})_t - \Delta u = 0,$$

with u denoting the water temperature. Observe that (St_2) can be viewed as a singular nonlinear parabolic equation of the form $[\beta(u)]_t - \Delta u = 0$ with an increasing function β.

Stefan problem and stopping times

It is natural to consider solutions for (St_1) generated by a particle system, which diffuses but stops its motion when it hits the ice. For $d = 1$ there has been several works in this direction, e.g. Chayes-Swindle (1996), Chayes-Kim (2008), Lacoin (2014), Delarue-Nadtochiy-Shkolnikov (2019).

We will show global existence for multi-dimensional solutions of (St_1), also based on Brownian particles with stopping times. Our novelty lies in the optimization structure of the stopping times, which turns out to be successful in achieving a stable and physically meaningful solutions of (St_1).

Stopping time

A **stopping time** τ of Brownian motion is, roughly speaking, a random time prescribed to satisfy a certain probabilistic condition
(that is the ”decision” of whether to stop at a given time t should only depend on the information given by the time t),
at which one stops a particle following the Brownian motion.

Skorokhod problem

For bounded, compactly supported probability measures μ and ν, we consider a **stopping time** τ of the Brownian motion such that

$$B_0 \sim \mu \text{ and } B_\tau \sim \nu, \text{ such that } E(\tau) < \infty.$$

Such τ exists if and only if μ satisfies *subharmonic order* with ν denoted by $\mu \leq_{SH} \nu$, that is, the solution to

$$-\Delta\varphi = \mu - \nu \text{ decay at infinity}$$

is non-negative. Roughly speaking this means that ν is more "spread out" then μ.

Optimal Skorokhod problem

For a cost $C(\tau) = E[\int_0^\tau L(t, B_t)dt]$, consider an optimal stopping time

$$\tau^* = \operatorname{argmin}\{C(\tau) : B_0 \sim \mu \text{ and } B_\tau \sim \nu\}.$$

For example, $L(t) = t$, then we minimize $E(\tau^2)$, which was the cost considered in Root (1969). Rost (1976) corresponds to $L(t) = -t$, thus maximizing $E(\tau^2)$.

If L is strictly monotone in time, then the optimal stopping time is the first hitting time of a space-time barrier set. Moreover, the optimal stopping time and the barrier is unique.

Root (1969), Rost (1976), McConnell ('91), Cox-Wang ('13),...

Beiglböck-Cox-Huesmann ('13), Ghoussoub-YH Kim-Palmer ('19)

Barrier for the optimal stopping time

We divide into the case L is strictly increasing (I) or decreasing (II) in time. The barriers are then given by $\{s(x) \leq t\}$ for (I) and $\{s(x) \geq t\}$ for (II), where $0 \leq s(x) \leq \infty$ is a measurable function.

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Question: For given μ, τ and ν, can we understand the barrier set associated with the stopping time in the frame of a free boundary problem that depends only on μ and ν?

Eulerian formulation

Theorem (Ghoussoub-Kim-Palmer 2019)

An optimal stopping time τ between μ and ν corresponds to a pair (η, ρ) solving the unique pair of solutions to the equation satisfying

$$\begin{cases} (\partial_t - \frac{1}{2}\Delta)\eta = -\rho; \\ \\ \rho(S) = 1 \text{ and } \eta(S) = 0; \\ \\ \int_{\mathbb{R}+} d\rho = \nu, \quad \eta(\cdot, 0) = \mu. \end{cases}$$

Here S is the barrier set associated with τ.

When S has nice boundaries, ρ must be concentrated on ∂S.

Heuristic discussions: Type (I)

Here the barrier set S is of the form $\{s(x) \leq t\}$, and ρ should yield ν when projected onto x-axis. Hence

$$\int_0^t \rho(x,\tau)d\tau = \nu\chi_{\{s(x)\leq t\}}, \text{ or } \rho(x,t) = \nu(\chi_{\{s(x)\leq t\}})_t.$$

The set $\{s(x) > t\}$ is also equal to the active mass region $\{\eta > 0\}$. Hence for costs of type (I) we can write

$$\rho = -\nu(\chi_{\{\eta>0\}})_t.$$

The Stefan problem

The (η, ρ) formulation thus leads us to the unstable (supercooled) Stefan problem, which can be written as

$$(\partial_t - \frac{1}{2}\Delta)\eta = +\nu(\chi_{\{\eta>0\}})_t$$

for costs of **type (I)**. Similarly it leads to the stable (melting) Stefan problem

$$(\partial_t - \frac{1}{2}\Delta)\eta = -\nu(\chi_{\{\eta>0\}})_t$$

for costs of **type (II)**.

This connection can be made rigorously, extending known results in one space dimension for the stable case (McConnell, 91). We suppose μ and ν are compactly supported and bounded.

Theorem

Let (η, ρ) correspond to the optimal stopping time τ from μ to ν. Suppose in addition that $\tau > 0$. Then η is a weak solution of the unstable Stefan problem for (I):

$$(St_1) \qquad (\eta - \nu\chi_{\{\eta>0\}})_t - \Delta\eta = 0, \quad \eta(\cdot, 0) = \mu$$

and the stable Stefan problem for (II):

$$(St_2) \qquad (\eta + \nu\chi_{\{\eta>0\}})_t - \Delta\eta = 0, \quad \eta(\cdot, 0) = \mu.$$

◦ Note that the support of μ may not equal $\{\eta(\cdot, 0^+) > 0\}$ with this formulation.

The general challenge for proving the theorem lies in the lack of regularity for the barrier (the function s is merely measurable). For instance one needs to show that

(a) ρ is concentrated on the barrier "boundary" $\{t = s(x)\}$;

(b) η is positive everywhere in the complement of the barrier.

Neither is clear with a measurable s, and one must argue carefully with the properties of stopping time as well as its optimality.

Reverse characterization: type (I)

Theorem

Let $\eta \in L^1_{t,x}$ be a weak solution of $(St1)_\nu$ with initial data μ. Let us define s and ρ by

$$s(x) := \sup\{t : \eta(x,t) > 0\} \text{ and}$$

$$\int\int \rho(x)\varphi(x,t)\,dxdt = \int \nu(x)\varphi(x,s(x))dx \quad \text{for any smooth } \varphi. \tag{1}$$

Then (η,ρ) is the Eulerian variables between $\eta_0 = \mu$ and $\tilde{\nu} := \nu\chi_{\{s<\infty\}}$ generated by the optimal stopping time with costs of type (I).

We would like to find solutions of the classical Stefan problem. This is especially an interesting task for the supercooled Stefan problem

$$(\eta - \chi_{\{\eta>0\}})_t - \Delta\eta = 0, \quad \eta(\cdot,0) = \mu,$$

which is well-known to be unstable. In view of the aforementioned results, this amounts to **finding ν as a characteristic function.**

We will add an additional optimization structure for the target, which will yield the optimal target ν as a characteristic function. The optimal target and its associated barrier thus provide us solutions of the Stefan problem.

Pro: It will enjoy stability and regularity properties due to the optimization procedure. In particular we suspect that for our solutions there is no jump of the free boundary over time.

Con: The solution constructed here will have its initial domain larger than that of μ.

Optimal stopping problem with free target

For a given absolutely continuous, compactly supported μ and for given f, we look for optimal ν such that the cost is minimized:

$$\inf_{\tau,\nu}\{C(\tau) : B_0 \sim \mu,\ B_\tau \sim \nu,\ \nu \leq f\}.$$

We have then the following, rather unexpected, universality.

Theorem

For given μ and f, the optimal target ν^* is the same for all costs of type I and type II.

Saturation of the optimal target

Theorem

With costs of type I:

$$\nu = f\chi_E + \mu|_F, \quad \textit{where } F = \{\tau = 0\} \textit{ and } E \cap F = \emptyset.$$

Note that $\mu|_F \leq f$.

Consequence of setting $f = 1$

Theorem

Let $\mu := (1+\eta_0)\chi_\Omega$ with $\eta_0 > 0$. Then the optimal target ν is a characteristic function, $\nu = \chi_E$, and the corresponding η solves

$$(St_1) \qquad (\eta - \chi_{\{\eta>0\}})_t - \Delta\eta = 0, \quad \eta(\cdot,0) = \mu.$$

Moreover

(a) *η is the unique weak solution of (St_1) that vanishes in finite time.*

(b) *The initial trace of the set $\{\eta > 0\}$ is $E = \{w > 0\}$, where $w \geq 0$ solves*

$$\chi_{\{w>0\}} - \Delta w = \mu. \tag{2}$$

Consequence of setting $f = \chi_{K^C}$, where $\mu = 0$ in K^C

Theorem

The optimal target is $\nu = \chi_\Sigma$, and η solves (St_1) with initial data μ. Moreover

(a) *η is the unique weak solution of (St_1) such that $\cap_{t>0}\{\eta(\cdot, t) > 0\} = K$.*

(b) *The initial trace is $E = \Sigma \cup K = \{u_\infty > 0\}$, where $u_\infty \geq 0$ solves*

$$\chi_{\{u_\infty > 0\} \setminus K} - \Delta u_\infty = \mu. \tag{3}$$

Stability of the optimal target

Proposition (Comparison)

Suppose $\mu_i \leq_{SH} \nu_i$ and $\nu_i \leq 1$ and let τ_i be the associated optimal stopping times for $i = 1, 2$. Suppose in addition that $\mu_1 \leq \mu_2$.

(a) *If ν_1 is the optimal target measure for μ_1, then $\tau_1 \leq \tau_2|_{\mu_1}$.*

(b) *If ν_2 is also optimal for μ_2, then $\nu_1 \leq \nu_2$.*

When $\mu_1 = \mu_2$, (a) states that the optimal target ν^* is the one with the least stopping time, given the constraint.

Proof of the universality

Theorem

For given μ, the optimal target ν^* is the same for all costs of type I and type II.

Proof.

Let ν_1 and ν_2 be optimal targets for costs C_1 and C_2, and let τ_1 and τ_2 be their respective stopping times. For C_1, consider the optimal stopping time τ_2' for ν_2. Then $\tau_1 \leq \tau_2'$, which means that $\nu_1 \leq_{SH} \nu_2$. Similarly we have $\nu_2 \leq_{SH} \nu_1$, which yields that $\nu_1 = \nu_2$. $\square$

Characterization of the optimal target when $f = 1$

We use universality of the optimal target $\nu = \chi_E$ for costs (I) and (II) with the measure $\mu = (1 + \eta_0)\chi_\Omega$ with $\eta_0 > 0$.
For type (II) costs, χ_Ω will instantly freeze and become part of ν, and the rest $\eta_0\chi_\Omega$ will spread with $\tau > 0$.

Then corresponding active mass η thus solves the stable Stefan problem

$$(\eta + \chi_{\{\eta>0\}})_t - \Delta\eta = 0$$

with initial data $\eta(\cdot, 0^+) = \eta_0\chi_\Omega$ and initial trace Ω.

We are interested in characterizing E, which equals the positive set of the total active mass, $\int_0^\infty \eta(\cdot, s)ds$.

Let $u(x,t) := \int_0^t \eta(x,s)ds$. Integrating the η-equation over time, we see that u solves

$$\begin{aligned} \eta + \chi_{\{\eta>0\}} - \Delta u &= u_t + \chi_{\{u>0\}} - \Delta u \\ &= \eta_0 \chi_\Omega + \chi_\Omega = \mu. \end{aligned}$$

As $t \to \infty$, η vanishes exponentially, and thus u tends to the equilibrium $u_\infty(x)$, which solves the obstacle problem

$$\chi_{\{u_\infty>0\}} - \Delta u_\infty = \mu,$$

and $E = \{u_\infty > 0\}$.

Thank you!