Kantorovich operators and their ergodic properties

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(based on joint work with Malcolm Bowles)

Kantorovich Initiative Seminar, March 2023

Outline

1. Non-linear Kantorovich operators in analysis.
2. 1-homogenous Kantorovich operators and "zero cost" balayage.
3. Kantorovich operators and Choquet functional capacities.
4. Duality: Kantorovich operators, linear transfers, optimal balayage.
5. Weak KAM solutions/operators associated to Kantorovich operators.
6. Deterministic and stochastic Fathi-Mather theory.
7. Deterministic and stochastic Ergodic optimization.

Formal definition of Kantorovich operators

A Markov operator is a map $T : C(Y) \to C(X)$ which is:

1. **positive**: if $g \geq 0$ then $Tg \geq 0$.
2. **linear**: $T(\lambda g_1 + \mu g_2) = \lambda T g_1 + \mu T^- g_2$.
3. **Markovian**: $T1 = 1$.
4. **continuous**: $g_n \to g$ in $C(Y)$,then $\lim_{n\to\infty} T g_n = Tg$.

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A backward Kantorovich operator $T^- : C(Y) \to USC(X)$ which is

1. **monotone increasing**, i.e., if $g_1 \leq g_2$, then $T^- g_1 \leq T^- g_2$.
2. **affine on the constants**, i.e., for any $c \in \mathbb{R}$ and $g \in C(Y)$ $T^-(g + c) = T^- g + c$.
3. **convex**, i.e., For any $\lambda \in [0, 1]$, we have

$$T^-(\lambda g_1 + (1 - \lambda) g_2) \leq \lambda T^- g_1 + (1 - \lambda) T^- g_2$$

4. **lower semi-continuous**, i.e., If $g_n \to g$, then $\liminf_{n\to\infty} T^- g_n \geq T^- g$.

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A forward Kantorovich operator is a map $T^+ : C(X) \to LSC(Y)$ that satisfies 1), 2), 3') (concave) and 4') (upper-semi-continuous).

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2. Plurisuperharmonic envelopes
$$U^- g(x) := \sup_{v \in \mathbb{R}^n} \left\{ \int_0^{2\pi} g(x + e^{i\theta} v) \frac{d\theta}{2\pi};\ x + \bar{\Delta} v \subset O \right\},$$
Iterates lead to
$$U^-_\infty g(x) = \sup \left\{ \int_0^{2\pi} g(P(e^{i\theta})) \frac{d\theta}{2\pi};\ P \text{ polynomial},\ P(\bar{\Delta}) \subset U,\ P(0) = x \right\}$$

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3. *Superharmonic envelope and Optimal stopping:*
$$U^- g(x) = \sup_{r \geq 0} \left\{ \int_B g(x + ry)\, dm(y);\ x + r\overline{B}\} \subset O \right\},$$
$$U^-_\infty g(x) := \sup \left\{ \mathbb{E}^x \Big[g(B_\tau) \Big];\ \tau \geq 0 \text{ stopping time}, \mathbb{E}^x[\tau] < +\infty \right\}.$$
The expectation $\mathbb{E}^x$ refers to Brownian motions $(B_t)_t$ starting at x.
4. The *filling scheme* for a Markov operator T, $U^- g = Tg^+ - g^-$.

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All these are positively 1-homogenous Kantorovich operators,
$$U(\lambda f) = \lambda U(f) \text{ for all } \lambda \geq 0.$$

1-homogenous Kantorovich operators

A typical 1-homogenous operator is

$$U^- g(x) := \sup\{\int_X g \, d\sigma; (x,\sigma) \in \mathcal{S}\},$$

where g is a reward function, and $\mathcal{S} \subset X \times \mathcal{P}(Y)$ is a "gambling house".

• For each x, $\mathcal{S}_x = \{\sigma \in \mathcal{P}(X); (x,\sigma) \in \mathcal{S}\} \neq \emptyset$ is the collection of distributions of gains available to a gambler having wealth x (Dubins-Savage, Dellacherie-Meyer).

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• When $X = Y$, a fair gambling houser is

$$U^- g(x) := \sup\{\int_X g\, d\sigma;\ \delta_x \prec_{\mathcal{A}} \sigma\},$$

where $\mathcal{A}$ is the cone of convex l.s.c. functions on X (convex compact):

$$\mu \prec_{\mathcal{A}} \nu \quad \text{iff} \quad \int_X \phi d\mu \leq \int \phi d\nu \text{ for all } \phi \in \mathcal{A}.$$

Convex order, Fair game, martingale, etc...
A very particular case, but somewhat characterizes 1-homogenous Kantorovich operators from Y to X.

A representation of 1-homogenous Kantorovich operators

Theorem: The following are equivalent:

1. T is a 1-homogenous Kantorovich operator from $C(Y)$ to $USC(X)$.
2. There exists a closed convex, stable under finite max, balayage cone $\mathcal{A}$ on the disjoint union $X \bigsqcup Y$ such that

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where for $(\mu, \nu) \in \mathcal{P}(X) \times \mathcal{P}(Y)$,

$\mu \prec_{\mathcal{A}} \nu$ iff $\int_X \phi_X d\mu \leq \int_Y \phi_Y d\nu$ for all $\phi \in \mathcal{A}$. (*Restricted balayage*).

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- The gambling house is then,

$$\mathcal{S} = \{(\mu,\nu) \in \mathcal{P}(X)\times\mathcal{P}(Y);\ \nu \leq U^{-}_{\#}\mu\} = \{(\mu,\nu) \in \mathcal{P}(X)\times\mathcal{P}(Y);\ \mu \prec_{\mathcal{A}} \nu\},$$

where $U^{-}_{\#}\mu(g) = \int_X U^{-}g\, d\mu$ for every $g \in C(Y)$.

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where $U^-_{\#}\mu(g) = \int_X U^- g \, d\mu$ for every $g \in C(Y)$.

• If $X = Y$, then iterating $Tu = u \vee U^- u$, leads to an idempotent backward Kantorovich operator U^-_∞ (i.e., $U^-_\infty \circ U^-_\infty = U^-_\infty$) and a balayage cone $\mathcal{A} \subset LSC(X)$ such that

$$U^-_\infty g(x) := \inf\{\phi(x); \phi \in -\mathcal{A}, \phi \geq g \text{ on } X\}.$$

Non-homogenous Kantorovich operators

1. Ergodic optimization of symbolic dynamics:

$$U^{-}g(x) := g \circ \sigma(x) - A(x),$$

A is a given potential and σ is a point transformation. Its iterates lead to minimizing the action $\mu \mapsto \int_X A d\mu$ among all σ-invariant measures μ.

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2. Optimal mass transport with cost function $c(x, y)$: on $X \times Y$.

$$U^- g(x) = \sup_{y \in Y}\{g(y) - c(x, y)\} \text{ resp., } U^+ f(y) = \inf_{x \in X}\{f(x) + c(x, y)\},$$

   is then a backward (resp., forward Kantorovich operator):
   (Brenier transport): $U^- g = -g^*$ resp., $U^+ f = (-f)^*$, ϕ^* is the Legendre transform.

3. Entropic regularization and Sinkhorn

$$T_\nu^- g(x) = \epsilon \log \int_Y e^{\frac{g(y) - c(x,y)}{\epsilon}} d\nu(y),$$

   where $\nu \in \mathcal{P}(X)$, as well as the composition $T_\nu^- \circ T_\mu^-$, where μ is another probability in $\mathcal{P}(X)$, whose iterates are the building clocks of the Sinkhorn algorithm.

Calculus of variations

If L is a Tonelli Lagrangian on TM

$$U^- g(x) := \sup \left\{ g(\gamma(1)) - \int_0^1 L(\gamma(s), \dot\gamma(s))\, ds; \gamma \in C^1([0,1), M); \gamma(0) = x \right\},$$

To a state g at time 1, it associates the initial state of the viscosity solution for the associated backward Hamilton-Jacobi equation,

$$\begin{cases} \partial_t V + H(x, \nabla_x V) &= 0 \ \text{ on } \ (0,1) \times M \\ V(1,x) &= g(x) \end{cases}$$

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All the above have corresponding forward Kantorovich operators.

One-sided Kantorovich operators

1. Stochastic control: One-sided Kantorovich operators appear in stochastic mass transfer problems. For example,

$$T^- g(x) = \sup_{X \in \mathcal{A}_{[0,1]}} \left\{ \mathbb{E}\left[g(X(1)) - \int_0^1 L(X(s), \beta_X(s,X))ds \; |X(0) = x \right] \right\},$$

where $\mathcal{A}_{[0,1]} := \{X : \Omega \to U; dX_t = \beta(t,X)\,dt + dW_t \quad \text{on } [0,1]\}$.
W_t is Weiner measure and the minimization is taken over all drifts β.

Under some assumptions on the Lagrangian L,
$T^- g = J_g(0,x)$ where J_g is the initial state of the backward second order Hamilton-Jacobi equation

$$\begin{cases} \partial_t J(t,x) + \frac{1}{2}\Delta J(t,x) + H\big(x, \nabla J(t,x)\big) & = 0 \text{ in } (0,1) \times \mathbb{R}^d, \\ J(1,x) & = g(x) \text{ on } \mathbb{R}^d. \end{cases}$$

Where H is the Hamiltonian associated to L.

Non-linear probability and potential theories?

1. A "non-linear potential theory"?: A cost $c : O \times O \to \mathbb{R} \cup \{+\infty\}$ is assigned to moving energy on a convex bounded domain in $\mathbb{R}^d$. The operator $U^- g(x) = u_{g,x}$, where $u_{g,x}$ is the unique minimiser of
$$\inf \Big\{ \int_O |\nabla u|^2 dy;\ u \geq g - c(x, \cdot),\ u \in H^1(O) \Big\},$$
is also a backward Kantorovich operator.

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2. Optimal stopping with cost: The operator $U^- g = J_g(0, \cdot)$, defined via the dynamic programming principle

$$U^- g(x) := \sup_{\tau \in \mathcal{R}^x} \Big\{ \mathbb{E}^x \Big[g(B_\tau) - \int_0^\tau L(s, B_s) ds \Big] \Big\},$$

If $t \to L(t, x)$ is decreasing, then U^- is an idempotent Kantorovich operator (i.e., $U^2 = U$).
$U^- g(x)$ is actually a "variational solution" at time 0, for the quasi-variational Hamilton-Jacobi-Bellman equation:

$$\min \left\{ \begin{array}{r} J(t,x) - g(x) \\ -\frac{\partial}{\partial t} J(t,x) - \frac{1}{2} \Delta J(t,x) + L(t,x) \end{array} \right\} = 0.$$

A characterization of general Kantorovich operators

Theorem: The following are equivalent:

1. T^- is a backward Kantorovich operator from $C(Y)$ to $USC(X)$.
2. There exists a l.s.c. cost functional $c : X \times \mathcal{P}(Y) \to \mathbb{R} \cup \{+\infty\}$ with $\sigma \mapsto c(x, \sigma)$ proper and convex for each $x \in X$, and a balayage cone $\mathcal{A} \in LSC(X \sqcup Y)$ such that

$$U^- g(x) := \sup\{\int_Y g d\sigma - c(x, \sigma) \,;\, \sigma \in \mathcal{P}(Y) \text{ and } \delta_x \prec_{\mathcal{A}} \sigma\},$$

A gambling house that charges fees: Unlike cost-free gambling houses, a gambler with wealth x, incurs a cost $c(x, \sigma)$ each time they choose a distribution of gains σ.

Kantorovich operators are functional capacities

Denote by $F_b(Y)$ (resp., $F^b(X)$) the class of functions on Y (resp., X) that are bounded above.

Theorem: Let $T : C(Y) \to USC(X)$ be a backward Kantorovich operator and let c be its cost. Then

1. T can be extended to a map from $F^b(Y)$ to $F^b(X)$ via the formula

$$Tg(x) = \sup\{\int_Y^* g d\nu - c(x,\nu);\ \nu \in \mathcal{P}(Y)\},$$

where $\int_Y^* g d\nu$ is the outer integral of g with respect to ν.

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$$T^- g(x) := \inf\{Th(x)\,;\ h \in C(Y),\ h \geq g\}.$$

3. If $T0$ is bounded below, then there is a constant k such that $T + k$ is a Choquet functional capacity that maps $F_+^b(Y)$ to $F_+^b(X)$.
4. If g is a K-analytic function that is bounded on Y, then

$$T^- g(x) := \sup\{T^- h(x)\,;\ h \in USC(Y)\,, h \leq g\}.$$

Kantorovich envelopes

A Choquet functional capacity is a map $T : F_+(Y) \to F_+(X)$ (The set of all non-negative functions valued in $\mathbb{R} \cup \{+\infty\}$) such that

1. T is monotone, i.e., $f \leq g \Rightarrow Tf \leq Tg$.
2. T maps $USC(Y)$ to $USC(X)$ and if $g_n, g \in USC(Y)$ and $g_n \downarrow g$, then $T^- g_n \downarrow T^- g$.
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3. If $g_n, g \in F_+(Y)$ with $g_n \uparrow g$, then $T^- g_n \uparrow T^- g$.

Theorem: Let $T : C(Y) \to USC_b(X)$ be a standard map. Then,

1. (Kantorovich envelope)

$$\underline{T}\, g(x) := \sup_{\sigma \in \mathcal{P}(Y)} \inf_{h \in C(Y)} \{ \int_Y (g - h)\, d\sigma + Th(x) \}$$

is the greatest Kantorovich operator S such that $S \leq T$ on $C(Y)$.

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2. (Choquet-Kantorovich envelope) If $T : F_+(Y) \to F_+(X)$ is a functional capacity, then

$$\overline{T} g(x) := \sup_{\sigma \in \mathcal{P}(Y)} \inf_{O \text{ open}} \{ \int_Y (g - \chi_O)\, d\sigma + T\chi_{\bar{O}}(x) \}$$

   is the greatest Kantorovich operator S such that $S(\chi_K) \leq T(\chi_K)$ for every compact K.

Wish Gustave Choquet was here

Let $T : F_+(Y) \to F_+(X)$ be a functional capacity.
Then there exists

1. a l.s.c. cost functional $c : X \times \mathcal{P}(Y) \to \mathbb{R} \cup \{+\infty\}$ with $\sigma \mapsto c(x, \sigma)$ proper and convex for each $x \in X$,
2. a balayage cone $\mathcal{A} \in LSC(X \sqcup Y)$

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and $\underline{T}$ is the greatest Choquet-Kantorovich functional capacity S such that $S(\chi_K) \leq T(\chi_K)$ for every compact K.

The secret: Linear Transfers

Let $\mathcal{T} : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and weak* lower semi-continuous on $\mathcal{M}(X) \times \mathcal{M}(Y)$. Write $D(\mathcal{T})$ for its domain.

- For $\mu \in \mathcal{P}(X)$, consider the partial maps $\mathcal{T}_\mu : \nu \to \mathcal{T}(\mu, \nu)$ on $\mathcal{P}(Y)$,
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1. $\mathcal{T}$ is a *backward linear Transfer*, if there exists an operator $T^- : C(Y) \to USC(X)$ such that for each $\mu \in \mathcal{P}(X)$, the Legendre transform of $\mathcal{T}_\mu$ on $\mathcal{M}(Y)$ satisfies:

$$\mathcal{T}_\mu^*(g) = \int_X T^- g(x)\, d\mu(x) \quad \text{for any } g \in C(Y).$$

$\mu \to \mathcal{T}_\mu^*$ is linear!

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$\mu \to \mathcal{T}_\mu^*$ is linear! We then have

$$\mathcal{T}(\mu, \nu) = \sup \Big\{ \int_Y g(y)\, d\nu(y) - \int_X T^- g(x)\, d\mu(x);\ g \in C(Y) \Big\}.$$

The secret: Linear Transfers

Let $\mathcal{T} : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and weak* lower semi-continuous on $\mathcal{M}(X) \times \mathcal{M}(Y)$. Write $D(\mathcal{T})$ for its domain.

- For $\mu \in \mathcal{P}(X)$, consider the partial maps $\mathcal{T}_\mu : \nu \to \mathcal{T}(\mu, \nu)$ on $\mathcal{P}(Y)$,
- For $\nu \in \mathcal{P}(Y))$, consider the partial map $\mathcal{T}_\nu : \mu \to \mathcal{T}(\mu, \nu)$ on $\mathcal{P}(X)$,

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$$\mathcal{T}(\mu, \nu) = \sup\Big\{ \int_Y g(y)\, d\nu(y) - \int_X T^- g(x)\, d\mu(x);\ g \in C(Y) \Big\}.$$

2. $\mathcal{T}$ is a *forward linear transfer*, if there exists an operator $T^+ : C(X) \to LSC(Y)$ such that for each $\nu \in \mathcal{P}(Y)$,

$$\mathcal{T}_\nu^*(f) = -\int_Y T^+(-f)(y)\, d\nu(y) \quad \text{for any } f \in C(X).$$

Hence,

$$\mathcal{T}(\mu, \nu) = \sup\Big\{ \int_Y T^+ f(y)\, d\nu(y) - \int_X f(x)\, d\mu(x);\ f \in C(X) \Big\}.$$

Optimal balayage transport

Optimal weak transport (Gozlan et al.) Let $c : X \times \mathcal{P}(Y) \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function such that for each $x \in X$, the function $\sigma \mapsto c(x, \sigma)$ is proper and convex. For $\mu \in \mathcal{P}(X)$ to $\nu \in \mathcal{P}(Y)$, it is

$$\mathcal{T}_c(\mu, \nu) := \inf_{\pi} \{ \int_X c(x, \pi_x) \, d\mu(x); \pi \in \mathcal{K}(\mu, \nu) \}, \text{where}$$

$$\mathcal{K}(\mu, \nu) = \left\{ \pi \in \mathcal{P}(X \times Y); \pi_X = \mu, \pi_Y = \nu, \pi(A \times B) = \int_A \pi_x(B) \, d\mu(x) \right\}$$

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Strassen.bis: If $\mathcal{A}$ is a balayage cone in $LSC(X \sqcup Y)$ and $\mu \prec_{\mathcal{A}} \nu$, then there exists $\pi \in \mathcal{K}_{\mathcal{A}}(\mu, \nu) = \{\pi \in \mathcal{K}(\mu, \nu); \delta_x \prec_{\mathcal{A}} \pi_x \quad \mu - a.e.\}$

Optimal balayage transport

$$\mathcal{B}_{c,\mathcal{A}}(\mu, \nu) = \begin{cases} \inf\{\int_X c(x, \pi_x) d\mu(x)\,;\, \pi \in \mathcal{K}_{\mathcal{A}}(\mu, \nu)\} & \text{if } \mu \preceq_{\mathcal{A}} \nu, \\ +\infty & \text{otherwise.} \end{cases}$$

A seminal duality

Theorem: The following are equivalent:

1. T^- is a backward Kantorovich operator from $C(Y)$ to $USC(X)$.
2. There is a backward linear transfer $\mathcal{T} : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathbb{R} \cup \{+\infty\}$ such that for all $\mu \in \mathcal{P}(X)$ and $g \in C(Y)$,

$$\mathcal{T}_\mu^*(g) = \int_X T^- g \, d\mu.$$

3. There exists an optimal balayage transport $\mathcal{T}_{c,\mathcal{A}}$, which is a backward linear transfer whose Kantorovich operator is given by

$$T^- g(x) := \sup\{\int_Y g d\sigma - c(x,\sigma)\, ; \, \sigma \in \mathcal{P}(Y), \delta_x \prec_{\mathcal{A}} \sigma\}.$$

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Corollary: T is positively homogenous iff $\mathcal{S}$ is a backward transfer set, i.e,
$$\mathcal{T}(\mu,\nu) = \begin{cases} 0 & \text{if } (\mu,\nu) \in \mathcal{S} \\ +\infty & \text{otherwise}, \end{cases}$$
is a zero-cost backward linear transfer, in which case
$$\mathcal{S} = \{(\mu,\nu) \in \mathcal{P}(X) \times \mathcal{P}(Y);\, \mu \prec_{\mathcal{A}} \nu\}.$$

Iterating a Kantorovich operator

Proposition (Iterations of Kantorovich operators)
Let $X_1, ..., X_n$ be n compact spaces, and for each $i = 1, ..., n$,
• $\mathcal{T}_i$ is a backward linear transfer on $\mathcal{P}(X_{i-1}) \times \mathcal{P}(X_i)$
• $T_i : USC(X_i) \to USC(X_{i-1})$ is the associated Kantorovich operator.
For $(\mu, \nu) \in \mathcal{P}(X_1) \times \mathcal{P}(X_n)$, define

$$\mathcal{T}_1 \star \mathcal{T}_2 ... \star \mathcal{T}_n(\mu, \nu) = \inf\{\mathcal{T}_1(\mu, \sigma_1) + \mathcal{T}_2(\sigma_1, \sigma_2) ... + \mathcal{T}_n(\sigma_{n-1}, \nu);\ \sigma_i \in \mathcal{P}(X_i), i = 1, n-1\}.$$

Then, $\mathcal{T} := \mathcal{T}_1 \star \mathcal{T}_2 ... \star \mathcal{T}_n$ is a linear backward transfer with a Kantorovich operator given by

$$T = T_1 \circ T_2 \circ ... \circ T_n.$$

Denote by $\mathcal{T}_n$ the linear transfer $\mathcal{T} \star \mathcal{T} \star ... \star \mathcal{T}$ by iterating n-times. The corresponding Kantorovich operator is then $T^n = T \circ T \circ ... \circ T$.

The Mañé constant of a Kantorovich operator

• **The self-transfer constant** of a backward linear transfer $\mathcal{T}$ and its Kantorovich operator $T: C(X) \to USC(X)$ is the -possibly infinite-

$$c(T) := \inf_{\mu \in \mathcal{P}(X)} \sup_{h \in C(X)} \{ \int_X (h - Th)\, d\mu = \inf_{\mu \in \mathcal{P}(X)} \mathcal{T}(\mu, \mu).$$

• If $c(T)$ is finite, then there exists $\bar{\mu} \in \mathcal{P}(X)$ such that $\mathcal{T}(\bar{\mu}, \bar{\mu}) = c(T)$. Such measures will be called **minimal measures**.

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• A **backward subsolution (resp., solution) for T at level $k \in \mathbb{R}$** is a function $g \in USC(X)$ so that

1. $Tg + k \leq g$ (resp., $Tg + k = g$) and
2. $\int_X g d\mu > -\infty$ for some minimal measure $\mu \in \mathcal{P}(X)$.

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• The **Mañé constant** c_0 is the supremum over all $k \in \mathbb{R}$ such that there exists a subsolution g for T at level k.

The self-transfer constant

Let $\mathcal{T} : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{+\infty\}$ be a backward linear transfer, T its associated backward Kantorovich operator. Then,

1. $c_0 = c(\mathcal{T})$
2. $c(\mathcal{T}) = \lim\limits_{n\to\infty} \frac{\inf\limits_{(\mu,\nu)\in\mathcal{P}(X)\times\mathcal{P}(X)} \mathcal{T}_n(\mu,\nu)}{n}$.
3. If $\bar{\mu}$ is a minimal measure, then for each $g \in C(X)$,

$$\lim_{n\to\infty} \frac{1}{n} \int_X T^n g \, d\bar{\mu} = -c(\mathcal{T}).$$

4. If $\mathcal{T}$ is bounded above on $\mathcal{P}(X) \times \mathcal{P}(X)$, then

$$\frac{\mathcal{T}_n(\mu,\nu)}{n} \to c(\mathcal{T}) \quad \text{uniformly on } \mathcal{P}(X) \times \mathcal{P}(X), \tag{1}$$

and for every $g \in C(X)$,

$$\frac{T^n g(x)}{n} \to -c(T) \quad \text{uniformly on } X. \tag{2}$$

Weak KAM operators associated to Kantorovich operators

Let $T : USC(X) \to USC(X)$ be a backward Kantorovich operator with a finite transfer constant $c(T)$.
Say that a Kantorovich operator $T_\infty : USC(X) \to USC(X)$ is *a backward weak KAM operator associated to T* if

1. T_∞ is idempotent.
2. $TT_\infty = T_\infty T$.
3. T_∞ maps $C(X)$ to the class of backward weak KAM solutions for T, i.e., for any $g \in C(X)$,

$$TT_\infty g + c(T) = T_\infty g.$$

The linear transfer associated to T_∞ is then

$$\mathcal{T}_\infty(\mu,\nu) = \sup_{g \in C(X)} \{\int_X g\, d\nu - \int_X T_\infty g\, d\mu\},$$

and will be called the **Peirls barrier associated to T**.

State of affairs

Three cases where we can prove the existence of a weak KAM operator associated to a Kantorovich opeartor:

1. When $\mathcal{T}$ is a weak*-continuous backward linear transfer.
2. When $c(T) = \inf\limits_{(\mu,\nu)\in\mathcal{P}(X)\times\mathcal{P}(X)} \mathcal{T}(\mu,\nu)$.
   For example, when T is 1-positively homogenous.
3. When $\mathcal{T}$ is of bounded oscillation.
   A linear transfer $\mathcal{T}$ *has bounded oscillation* if

$$\limsup_{n\to\infty}\{nc(\mathcal{T}) - \inf_{\mathcal{P}(X)\times\mathcal{P}(X)} \mathcal{T}_n\} < +\infty.$$

For example, when $\mathcal{T}$ is bounded above.

Ergodic properties of continuous linear transfers

Let $\mathcal{T}$ be a weak*-continuous backward linear transfer with backward Kantorovich operator T. Then, there exists a backward weak KAM operator $T_\infty^- : C(X) \to C(X)$ with a corresponding backward linear transfer $\mathcal{T}_\infty$ so that

1. $(\mathcal{T}_n - nc(\mathcal{T})) \star \mathcal{T}_\infty = \mathcal{T}_\infty = \mathcal{T}_\infty \star (\mathcal{T}_n - nc(\mathcal{T}))$ for every $n \in \mathbb{N}$.
2. For every $\mu, \nu \in \mathcal{P}(\mathcal{X})$, we have

$$\sup\left\{\int_X T_\infty^- g d(\nu - \mu)\,;\, g \in C(X)\right\} \leq \mathcal{T}_\infty(\mu,\nu) \leq \liminf_{n\to\infty}(\mathcal{T}_n(\mu,\nu) - nc(\mathcal{T})).$$

3. $\mathcal{A} := \{\sigma \in \mathcal{P}(X); \mathcal{T}_\infty(\sigma,\sigma) = 0\}$ contains all minimal measures of $\mathcal{T}$.
4. If $\mathcal{T}$ is also a forward linear transfer, then there exists conjugate functions ψ_0, ψ_1 for $\mathcal{T}_\infty$ in the sense that

$$\psi_0 = T_\infty^- \psi_1 \quad \psi_1 = T_\infty^+ \psi_0,$$

such that

$$T^- \psi_0 + c = \psi_0, \quad T^+ \psi_1 - c = \psi_1,$$

and

$$\int_X \psi_0 d\mu = \int_X \psi_1 d\mu \text{ for every } \mu \in \mathcal{A}.$$

Semi-groups of Kantorovich operators

Let $\{\mathcal{T}_t\}_{t\geq 0}$ be a family of backward linear transfers on $\mathcal{P}(X)\times\mathcal{P}(X)$ with associated Kantorovich operators $\{T_t\}_{t\geq 0}$,

(H0) $\{\mathcal{T}_t\}_{t\geq 0}$ is a semi-group for inf-convolution: $\mathcal{T}_{t+s}=\mathcal{T}_t\star\mathcal{T}_s$ $(s,t\geq 0)$

(H1) For every $t>0$, the transfer $\mathcal{T}_t$ is weak*-continuous.

(H2) For any $\epsilon>0$, $\{\mathcal{T}_t\}_{t\geq\epsilon}$ has common modulus of continuity $\delta(\epsilon)$.

Example: $A_t(x,y)$ be a semi-group of equicontinuous cost functions on $X\times X$, that is

$$A_{t+s}(x,y)=A_t\star A_s(x,y):=\inf\{A_t(x,z)+A_s(z,y); z\in X\},$$

and the associated optimal mass transports

$$\mathcal{T}_t(\mu,\nu)=\inf\{\int_{X\times X}A_t(x,y)d\pi(x,y)\,;\,\pi\in\mathcal{K}(\mu,\nu)\}.$$

$(\mathcal{T}_t)_t$ is then a semi-group of linear transfers and there is a backward and forward linear transfer $\mathcal{T}_\infty$, and weak KAM operators T_∞^-, T_∞^+ such that:

Weak KAM theories

If $c := c((\mathcal{T}_t)_t) := \lim\limits_{t\to\infty} \frac{\inf_{\mu,\nu\in\mathcal{P}(X)} \mathcal{T}_t(\mu,\mu)}{t}$. Then

1. $c = \min\{\int_{X\times X} A_1(x,y)d\pi; \pi \in \mathcal{P}(X\times X), \pi_1 = \pi_2\}$
2. $A_\infty(x,y) := \liminf\limits_{t\to\infty}(A_t(x,y) - ct)$ is continuous on $X \times X$, and
   - $\mathcal{T}_\infty(\mu,\nu) = \mathcal{T}_{A_\infty}(\mu,\nu) := \inf\{\int_{X\times X} A_\infty(x,y)d\pi(x,y)\,;\, \pi \in \mathcal{K}(\mu,\nu)\}$,
   - $T_\infty^- f(x) = \sup\{f(y) - A_\infty(x,y)\,;\, y \in X\}$, $T_\infty^+ f(y) = \inf\{f(x) + A_\infty(x,y)\,;\, x \in X\}$.
3. The minimizing measures in (1) are all supported on the set
$$D := \{(x,y) \in X \times X\,;\, A_1(x,y) + A_\infty(y,x) = c\}.$$
4. There exists conjugate functions u^-, u^+ for $\mathcal{T}_\infty$ in the sense that
$$u^-(x) = \sup\{u^+(y) - A_\infty(x,y)\,;\, y \in X\}, \quad u^+(y) = \inf\{u^-(x) + A_\infty(x,y)\,;\, x \in X\},$$
$$T_t^- u^- + ct = u^-, \quad T_t^+ u^+ - ct = u^+) \text{ for all } t \geq 0.$$
$$u^-(x) = u^+(x) \text{ whenever } A_\infty(x,x) = 0.$$

Weak KAM solutions in Lagrangian dynamics

Let L be a time-independent *Tonelli Lagrangian* on a compact Riemanian manifold M, and consider $\mathcal{T}_t$ to be the cost minimizing transport

$$\mathcal{T}_t(\mu,\nu) = \inf\{\int_{M\times M} A_t(x,y)d\pi(x,y)\,;\,\pi\in\mathcal{K}(\mu,\nu)\}, \text{where}$$

$$A_t(x,y) := \inf\{\int_0^t L(\gamma(s),\dot\gamma(s))ds\,;\,\gamma\in C^1([0,t];M);\gamma(0)=x,\gamma(t)=y\}.$$

The backward (forward) Lax-Oleinik semi-group is defined for $t>0$, via

$$S_t^- u(x) = \sup\{u(\gamma(t)) - \int_0^t L(\gamma(s),\dot\gamma(s))ds\,;\,\gamma\in C^1([0,t];M),\gamma(0)=x\},$$

$$S_t^+ u(x) := \inf\{u(\gamma(0)) + \int_0^t L(\gamma(s),\dot\gamma(s))ds\,;\,\gamma\in C^1([0,t];M),\gamma(t)=x\}.$$

A function $u\in C(M)$ is said to be a *backward (resp., forward) weak KAM solution* if $S_t^- u + ct = u$ (resp., $S_t^+ u - ct = u$) for all $t\geq 0$.

Theorem: There exists a unique constant $c \in \mathbb{R}$ such that:

1. **(Fathi)** There exists weak KAM solutions, i.e., $u_- : M \to \mathbb{R}$ (resp. u_+) such that $S_t^- u_- + ct = u_-$ (resp. $S_t^+ u_- - ct = u_-$) for $t \geq 0$.

2. **(Bernard-Buffoni)** Let $A_\infty(x, y) := \liminf_{t\to\infty} A_t(x, y) - tc$ denotes the *Peierls barrier function*. Then,

$$\inf\{\int_{M\times M} A_\infty(x, y)d\pi(x, y)\,;\, \pi \in \mathcal{K}(\mu, \nu)\} = \sup_{u_+, u_-}\{\int_M u_+ d\nu - \int_M u_- d\mu\},$$

   and

$$u_+ = u_- \text{ on } \mathcal{A} := \{x \in M\,;\, A_\infty(x, x) = 0\}.$$

3. **(Bernard-Buffoni)** $c = \min_\pi \int_{M\times M} A_1(x, y)d\pi(x, y)$, over all $\pi \in \mathcal{P}(M \times M)$ with equal first and second marginals.
   The minimizing measures are all supported on

$$\mathcal{D} := \{(x, y) \in M \times M\,;\, A_1(x, y) + A_\infty(y, x) = c\}.$$

4. **(Mather)** $c = \inf_m \int_{TM} L(x, v)dm(x, v)$ over all measures $m \in \mathcal{P}(TM)$ which are invariant under the Euler-Lagrange flow.

5. **(Fathi)** A continuous function $u : M \to \mathbb{R}$ is a viscosity solution of

$$H(x, \nabla u(x)) = c$$

   if and only if it is Lipschitz and u is a backward weak KAM solution.

Stochastic (2d order Fathi-Mather theory

(Gomez, Mikami) $M = \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ being the d-dimensional flat torus, $(\Omega, \mathcal{F}, \mathcal{P})$ a complete probability space with normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$,
Let $\mathcal{A}_{[0,t]}$ be the set of continuous semi-martingales $X : \Omega \times [0,t] \to M$ such that for some drift $\beta_X : [0,t] \times C([0,t]) \to \mathbb{R}^d$,

$$dX_t = \beta(t, X)dt + dW_t$$

where W_t is an M-valued Brownian motion.

$$\mathcal{T}_t(\mu, \nu) := \inf \left\{ \mathbb{E} \int_0^t L(X(s), \beta_X(s, X))\underset{\cdot}{s}\,;\, X(0) \sim \mu, X(t) \sim \nu, X \in \mathcal{A}_{[0,t]} \right\},$$

is then a backward linear transfer, with Kantorovich operator

$$T_t f(x) := \sup_{X \in \mathcal{A}_{[0,t]}} \left\{ \mathbb{E}\left[f(X(t)) - \int_0^t L(X(s), \beta_X(s, X))\underset{\cdot}{s} \,\Big|\, X(0) = x \right] \right\}.$$

$$\mathcal{N}_0 := \left\{ m \in \mathcal{P}(TM)\,;\, \int_{TM} [\frac{1}{2}\Delta(x)\phi + v \cdot \nabla\phi(x)]\, dm(x, v) = 0 \text{ for all } \phi \in C^2(M) \right\}.$$

(Euler-Lagrange Flow invariant measures on phase space)

1. $c := \inf\{\mathcal{T}_1(\mu, \mu)\,;\, \mu \in \mathcal{P}(M)\} = \inf\{\int_{TM} L(x, v)\underset{\cdot}{m}(x, v);\ m \in \mathcal{N}_0\}$.
   Infimum is attained by a measure $\bar{m}$, a **stochastic Mather measure.**
   Its projection $\mu_{\bar{m}}$ on $\mathcal{P}(M)$ is a minimiser for $\mathcal{T}_1$.
2. There exists *backward weak KAM solutions* $T_t u + ct = u$ for $t \geq 0$, $u \in C(M)$,
3. The backward weak KAM solutions are exactly the viscosity solutions of the stationary Hamilton-Jacobi-Bellman equation $\frac{1}{2}\Delta u + H(x, D_x u) = c$.

Symbolic dynamics (Garibaldi-Lopez)

Let M is an $r \times r$ transition matrix, whose $\{0,1\}$ entries specify allowable transitions.

$$\Sigma = \{x \in \{1, ..., r\}^{\mathbb{N}} \; ; \; M(x_i, x_{i+1}) = 1, \forall i \geq 0\}$$

the set of admissible words, and its dual

$$\Sigma^* = \{y \in \{1, ..., r\}^{\mathbb{N}} \; ; \quad M(y_{i+1}, y_i) = 1, \forall i \geq 0\}$$

and consider

$$\hat{\Sigma} = \{(y, x) \in \Sigma^* \times \Sigma \; ; \; M(y_0, x_0) = 1\}.$$

Assume $\Sigma_x^* := \{y \in \Sigma^* \; ; \; (y, x) \in \hat{\Sigma}\} \neq \emptyset$, $\forall x \in \Sigma$.
Consider the time-evolution map $\sigma : \Sigma \to \Sigma$ and $\tau : \hat{\Sigma} \to \Sigma$ defined as

$$\sigma(x_0, x_1, ...) = (x_1, x_2, ...) \quad \text{and} \quad \tau(y, x) = (y_0, x_0, x_1, ...).$$

The set of holonomic probability measures is

$$\mathcal{M}_0(\hat{\Sigma}) := \left\{\mu \in \mathcal{P}(\hat{\Sigma}) \; ; \; \int_{\hat{\Sigma}} f(\tau_y(x)) - f(x) \; d\mu(y, x) = 0\right\}.$$

Theorem: Given $A \in C(\hat{\Sigma})$, define $\beta_A := \max_{\hat{\mu} \in \mathcal{M}_0(\hat{\Sigma})} \int_{\hat{\Sigma}} A(y, x) d\hat{\mu}(y, x)$. Then

$$\beta_A = \inf_{f \in C(\Sigma)} \max_{(y,x) \in \hat{\Sigma}} \{A(y, x) + f(x) - f(\tau_y(x))\}.$$

There exists $u \in USC(\Sigma)$ such that

$$\inf_{y \in \Sigma_x^*} \{u(\tau_y(x)) - A(y, x) + \beta(A)\} = u(x) \quad \forall x \in \Sigma.$$

Thank you