Initial value problems viewed as generalized optimal transport problems with matrix-valued density fields

Yann Brenier CNRS, DMA/ENS, 45 rue d'Ulm, FR-75005 Paris,
in association with the CNRS-INRIA team "MOKAPLAN".

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It turns out that this method also applies to some parabolic equations: porous medium, viscous Hamilton-Jacobi and incompressible Navier-Stokes.

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which is nothing but the macroscopic limit of the properly rescaled (deterministic) system of particles:

$$\frac{dX_k}{dt} = \epsilon^{-1} \sum_{j=1,N} (X_k - X_j) \exp\left(-\frac{|X_k - X_j|^2}{\epsilon}\right),$$

$$u(t,x) \sim \frac{1}{N} \sum_{j=1,N} \delta(x - X_j(t)), \quad 1/N << \epsilon^d << 1.$$

cf. P.-L. Lions, S. Mas-Gallic 2001 and ...A. Figalli, R. Philipowski 2008 .

A weird minimization problem.

We start with the rather absurd problem of minimizing the time integral of the "entropy"

$$\int_Q u^2(t,x)dxdt, \quad Q = [0,T] \times \mathbb{T}^d,$$

among weak solutions ot the QPME

$$\partial_t u = \Delta u^2/2, \quad u = u(t,x) \in \mathbb{R}, \quad t \geq 0, \quad x \in \mathbb{T}^d.$$

with a given initial condition $u_0 \geq 0$ in $L^\infty(\mathbb{T}^d)$.

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where the only constraints are:
i) for test function ϕ to be smooth and vanish at $t = T$;
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This problem admits an interesting concave relaxation:

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$$\sup_{\phi} \int_Q \left(-\frac{(\partial_t \phi)^2}{1 - \Delta\phi} + 2u_0 \partial_t \phi \right), \quad \Delta\phi \leq 1, \quad \phi(T, \cdot) = 0.$$

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Setting $q = \partial_t\phi$, $\sigma = 1 - \Delta\phi$, we get: $J(u_0) =$

$$\sup_{\sigma,q} \int_Q \left(-\frac{q^2}{\sigma} + 2u_0\ q\right), \quad \partial_t\sigma + \Delta q = 0, \quad \sigma(T,\cdot) = 1$$

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s.t.

$$\partial_t \sigma + \Delta q = 0, \quad \sigma(T,\cdot) = 1,$$

is (at least as $d = 1$) almost the same as the recent formulation "à la Benamou-Brenier" proposed by Huesmann and Trevisan for the time-discrete martingale optimal transport problem.

(See also Ghoussoub-Kim.)

Main result: there is no duality gap!

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Let us try to find a solution ϕ to the concave optimization problem just by solving the final VP

$$\partial_t \phi = (1 - \Delta\phi)u, \quad \phi(T, \cdot) = 0,$$

i.e., for $\alpha = 1 - \Delta\phi$: $\partial_t \alpha + \Delta(\alpha u) = 0, \quad \alpha(T, \cdot) = 1.$

From Aronson-Bénilan, we deduce $\alpha(t, x) \geq (t/T)^\kappa$.

Proof. (Assuming u to be smooth) we have

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So, $\log A(T) - \log A(t) \leq \kappa(\log T - \log t)$, and therefore

$A(t) \geq (t/T)^\kappa$ (since $A(T) = 1$). End of proof.

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(using $\partial_t \phi = (1 - \Delta\phi) u$) which shows that ϕ is optimal since $J(u_0) \geq j = \int_Q u^2 \geq I(u_0) \geq J(u_0)$.

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Minimize $\int_Q |B|^2$ among all *weak* solutions of

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$$\inf_{\rho, q} \int_Q q \cdot B_0 + \frac{|q - \epsilon \nabla \rho|^2}{2\rho}$$

where the fields $\rho \geq 0$, $q \in \mathbb{R}^d$ are constrained by

$$\partial_t \rho + \nabla \cdot q = 0, \quad \rho(T, \cdot) = 1.$$

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$$\inf_{\rho,q} \int_Q \frac{|q|^2 + \epsilon^2 |\nabla \rho|^2}{2\rho}$$

$$+ \int_D \rho(0,\cdot)(\epsilon \log \rho(0,\cdot) + \phi_0), \quad \text{s.t.} \quad \partial_t \rho + \nabla \cdot q = 0, \quad \rho(T,\cdot) = 1,$$

i.e. as a variant of the "Schrödinger problem",

a noisy version of the optimal transport problem with quadratic cost, intensively studied in the recent years, after Ch. Léonard, e.g. in the CNRS-INRIA MOKAPLAN team (mostly for numerical purposes), and very recently by A. Baradat, and L. Monsaingeon.

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$$\inf_{M,q} \int_Q (q - \epsilon \nabla \cdot M) \cdot M^{-1} \cdot (q - \epsilon \nabla \cdot M) - 2q \cdot v_0$$

where $Q = [0, T] \times \mathbb{T}^d$, the matrix-valued field $M = M^T \geq 0$ and the vector field q being subject to

$$\partial_t M + \nabla q + \nabla q^T = 2D^2 \Delta^{-1} \nabla \cdot q, \quad M(T, \cdot) = I_d.$$

Few remarks

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1) The Schrödinger problem (1931) is closely related to the Schrödinger equation (1925), which can be solved by looking at critical points (ρ, q) of the following action (featuring a crucial change of sign):

$$\int_Q \frac{|q|^2 - |\nabla\rho|^2}{2\rho} \quad \text{s.t.} \quad \partial_t \rho + \nabla \cdot q = 0,$$

through the Madelung transform (1926):

$$\psi = \sqrt{\rho}\, e^{i\theta}, \quad q = \rho \nabla\theta.$$

Few remarks (continued)

2) The optimization problem we obtained from the NS equations can be seen as a (very special) example of a matrix-valued Optimal Transport problem (*), for which we may refer to a collection of works by Tryphon Georgiou and coll., and a recent paper by Y.B. and Dmitry Vorotnikov (SIMA 2020).

(*) due to the special structure of its time-boundary conditions, the NS optimization problem more precisely corresponds to a matrix-valued Mean-Field Game problem.

Few remarks (continued)

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very roughly similar to the 4D-Einstein action (*)

$$(\Gamma^m_{ij}\, g^{ij}\, \Gamma^k_{km} - \Gamma^m_{ik}\, g^{ij}\, \Gamma^k_{jm})\sqrt{-\det g}$$

where g_{ij} is Lorentzian of inverse g^{ij} and connection

$$\Gamma^i_{jk} = g^{im}(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{kj})/2.$$

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* Note that General Relativity has been recently related to Optimal Transportation

(in particular by R. McCann arXiv:1808.01536, A. Mondino, S. Suhr arXiv:1810.13309).

Let us finish with the inviscid Burgers equation

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In that case, the concave maximization problem reads

$$\sup_{(\rho,q)} \left\{ \int_{[0,T]\times\mathbb{T}} -\frac{q^2}{2\rho} - qu_0 \mid \partial_t\rho + \partial_x q = 0, \;\; \rho(T,\cdot) = 1 \right\}.$$

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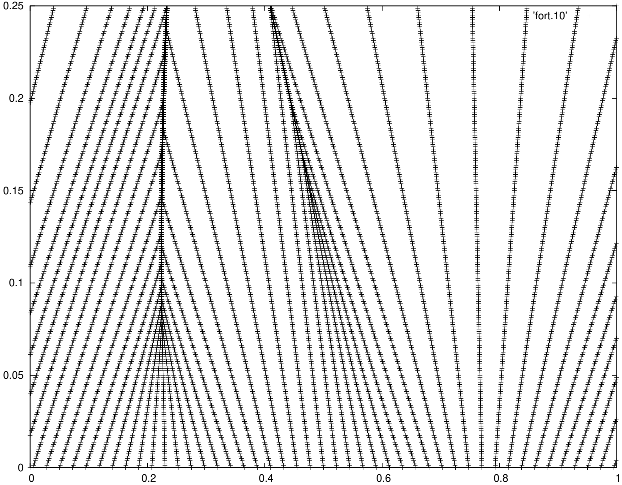
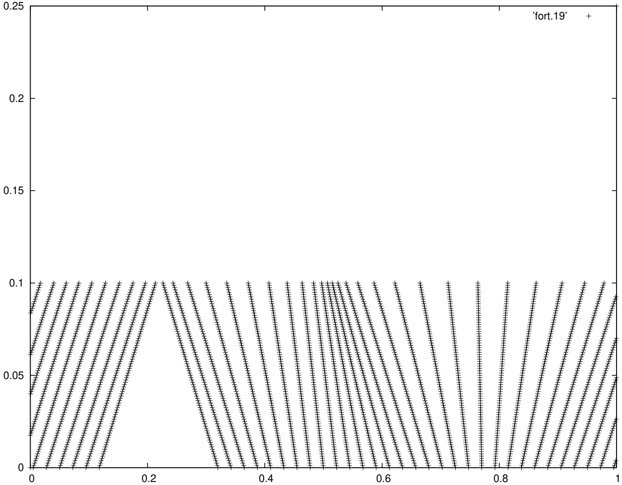
As shown in CMP 2018, for arbitrarily large T, we recover, through this problem, the correct "entropy solution" à la Kruzhkov, but only at time T and (surprisingly enough) not for $t < T$, once shocks form!



Inviscid Burgers equation : $\partial_t u + \partial_x(u^2/2) = 0$, $u = u(t,x)$, $x \in \mathbb{R}/\mathbb{Z}$, $t \geq 0$.
Formation of two shock waves. (Vertical axis: $t \in [0, 1/4]$, horizontal axis: $x \in \mathbb{T}$.)



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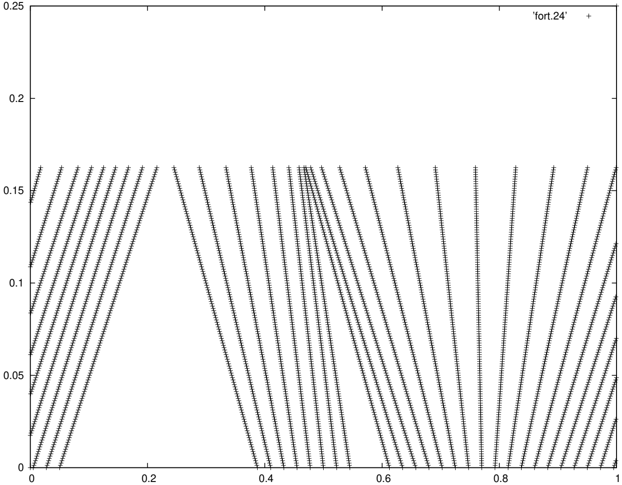
Recovery of the solution at time T=0.1 by convex optimization.
Observe the formation of a first vacuum zone as the first shock has formed.



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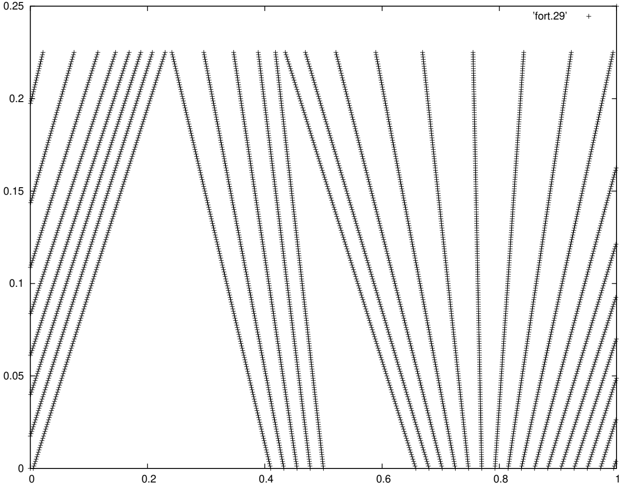
Recovery of the solution at time T=0.16 by convex optimisation.
Observe the formation of a second vacuum zone as the second shock has formed.



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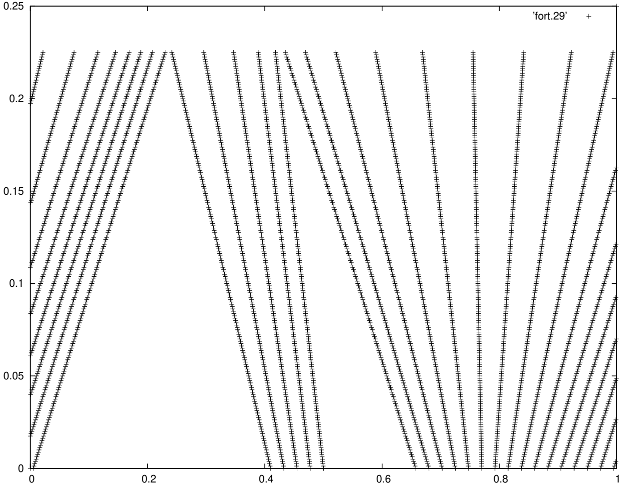
Recovery of the solution at time T=0.225 by convex optimisation.
Observe the extension of the two vacuum zones.



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THANKS FOR YOUR ATTENTION!

IV. Entropic systems of conservation laws

$$\partial_t U + \nabla \cdot (F(U)) = 0, \;\; U = U(t,x) \in \mathcal{W} \subset \mathbb{R}^m, \;\; x \in \mathbb{T}^d,$$

involve a strictly convex "entropy" $\mathcal{E} : \mathcal{W} \to \mathbb{R}$ (where $\mathcal{W}$ is convex) and an "entropy flux" $\mathcal{Z} \in \mathcal{W} \to \mathbb{R}^d$, such that each smooth solution U satisfies the extra conservation law $\partial_t(\mathcal{E}(U)) + \nabla \cdot (\mathcal{Z}(U)) = 0.$

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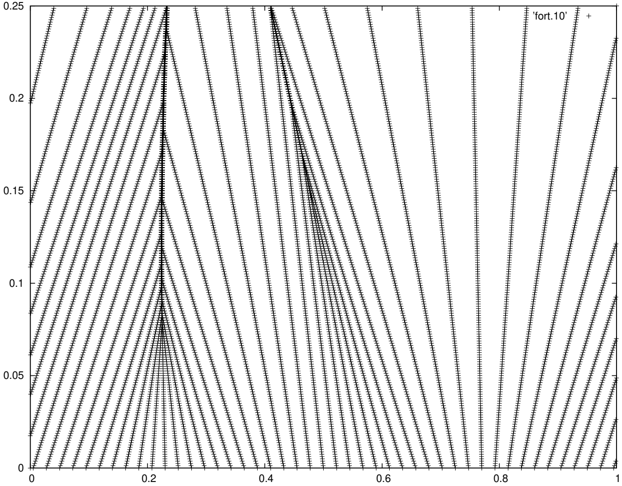
A typical example is the (barotropic) Euler system, where $U = (\rho, q) \in \mathbb{R}_+ \times \mathbb{R}^d$, with entropy $\mathcal{E}(\rho, q) = \frac{|q|^2}{2\rho} + \Phi(\rho)$ and pressure $p(\rho) = \int_0^\rho s\Phi''(s)ds$.



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The problem is not trivial since there may be many weak solutions starting from U_0 which are not entropy-preserving (by "convex integration" à la De Lellis-Székelyhidi).

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where $G(E,B) = \sup_{V \in \mathcal{W} \subset \mathbb{R}^m} E \cdot V + B \cdot F(V) - \mathcal{E}(V),$

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Observe that G is automatically convex.

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Main results (Y.B. CMP 2018)

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Theorem 2: For the Burgers equation, all entropy solutions can be recovered, for arbitrarily large T.

(*) more precisely if, $\forall\ t, x, V \in \mathcal{W}$, $\mathcal{E}''(V) - (T-t)F''(V) \cdot \nabla(\mathcal{E}'(U(t,x))) > 0$.

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$$u = \partial_t\sigma + \partial^i w_i,\ \ Q_i = \partial_t w_i + \partial_i\sigma,\ \ M_{ij} = \delta_{ij} - \partial_i w_j - \partial_j w_i,$$

where σ and w must vanish at $t = T$.